EXERCISE SOLUTIONS, LECTURES 8-14

Contents

8. Derivatives and integrals of vector functions

Exercise 1. Find the domain of the vector function and determine whether they are closed and/or bounded. Determine whether the domain is compact or not.

(1) $\vec{r}(t) = \langle \sqrt{t}, t^2 + 1, \ln(t) \rangle$
(2) $\vec{r}(t) = \langle \sin(e^t), \sqrt{e^t}, \sqrt{-t} \rangle$
(3) $\vec{r}(t) = \langle \sqrt{2+t}, \sqrt{2-t}, \frac{1}{t+4} \rangle$

Solution.

(1) Note that $\sqrt{t}$ is defined for $t \geq 0$, $t^2 + 1$ is defined everywhere, and $\ln(t)$ is defined for $t > 0$. So, for them to be simultaneously defined, $t > 0$. This is not closed, not bounded, so not compact.
(2) Note that $\sin(e^t)$ is defined everywhere, $\sqrt{e^t}$ is defined everywhere, and $\sqrt{-t}$ is defined for $-t \geq 0$, or $t \leq 0$. So, for them to be simultaneously defined, $t \leq 0$. This is closed, but not bounded, so not compact.
(3) Note that $\sqrt{2+t}$ is defined for $2 + t \geq 0$, or $t \geq -2$. Also, $\sqrt{2-t}$ is defined for $2 - t \geq 0$, or $2 \geq t$. Finally, $\frac{1}{t+4}$ is defined for $t + 4 \neq 0$, or $t \neq -4$. So, for them to be simultaneously defined, $-2 \leq t \leq 2$. This is closed and bounded, so compact.

□

Exercise 2. Find the limit, or explain why it does not exist.

(1) $\lim_{t\to 0} \langle t^2 + 1, \frac{1}{t+1}, e^t \rangle$.
(2) $\lim_{t\to 0} \langle \frac{\sin(t)}{t}, \frac{e^t - 1}{t}, t \rangle$.
(3) $\lim_{t\to\infty} \langle \frac{1}{t}, t + 1, e^t \rangle$.
(4) $\lim_{t\to\infty} \langle \frac{t^2+1}{2t^2+t}, \frac{\ln(t)}{t}, \frac{t+1}{t^2+1} \rangle$.
(5) $\lim_{t\to 1} \langle \frac{t-1}{t^2-1}, \frac{e^t - 1}{t}, \frac{1}{t+3} \rangle$.

Solution.

(1) Plug $t = 0$ and get $\langle 1, 1, 1\rangle$. This is possible because the component functions are all continuous at $t = 0$.

(2) Note by L'Hopital (because these are $\frac{0}{0}$)

$$\lim_{t\to 0} \frac{\sin(t)}{t} = \lim_{t\to 0} \frac{\cos(t)}{1} = \cos(0) = 1, \quad \lim_{t\to 0} \frac{e^t - 1}{t} = \lim_{t\to 0} \frac{e^t}{1} = e^0 = 1.$$

So the limit is $\langle 1, 1, 0\rangle$.

(3) Since $\lim_{t\to\infty} e^t$ doesn't exist, the limit doesn't exist.

(4) Note by L'Hopital (because these are $\frac{\infty}{\infty}$)

$$\lim_{t\to\infty} \frac{t^2 + 1}{2t^2 + t} = \lim_{t\to\infty} \frac{2t}{4t + 1} = \lim_{t\to\infty} \frac{2}{4} = \frac{1}{2},$$
$$\lim_{t\to\infty} \frac{\ln(t)}{t} = \lim_{t\to\infty} \frac{\frac{1}{t}}{1} = 0,$$
$$\lim_{t\to\infty} \frac{t + 1}{t^2 + 1} = \lim_{t\to\infty} \frac{1}{2t} = 0,$$

so the limit is $\langle \frac{1}{2}, 0, 0\rangle$.

(5) Note that for $t \neq 1$, $\frac{t-1}{t^2-1} = \frac{1}{t+1}$. So the limit is just $\lim_{t\to 1}\langle \frac{1}{t+1}, \frac{e^t-1}{t}, \frac{1}{t+3}\rangle = \langle \frac{1}{2}, \frac{e-1}{1}, \frac{1}{1+3}\rangle = \langle \frac{1}{2}, e - 1, \frac{1}{4}\rangle$.

□

Exercise 3. Find $\vec{r}\,'(t)$.

(1) $\vec{r}(t) = \langle t^5, -2t\rangle$
(2) $\vec{r}(t) = e^{3t}\vec{i} + \cos(t)\vec{j} + \frac{1}{1+t}\vec{k}$
(3) $\vec{r}(t) = \langle \sqrt{t + 2}, 2, \frac{1}{t^3}\rangle$

Solution.

(1) $\vec{r}\,'(t) = \langle 5t^4, -2\rangle$.
(2) $\vec{r}\,'(t) = 3e^{3t}\vec{i} - \sin(t)\vec{j} - \frac{1}{(1+t)^2}\vec{k}$.
(3) $\vec{r}\,'(t) = \langle \frac{1}{2\sqrt{t+2}}, 0, -\frac{3}{t^4}\rangle$.

□

Exercise 4. Find the unit tangent vector $\vec{T}(t)$ at the point with the given value of the parameter t.

(1) $\vec{r}(t) = \langle \sin t, \cos 2t, \cos t + \sin t\rangle, \quad t = 0.$
(2) $\vec{r}(t) = \langle e^{-t}, t^2 + 2t, e^t\rangle, \quad t = 0.$
(3) $\vec{r}(t) = \langle \ln(t + 1), \cos(t), \frac{1}{t^2-3}\rangle, \quad t = 0.$

Solution.

(1) We have

$$\vec{r}\,'(t) = \langle \cos t, -2\sin 2t, -\sin t + \cos t\rangle, \quad \vec{r}\,'(0) = \langle 1, 0, 1\rangle.$$

So $\vec{T}(0) = \frac{\vec{r}\,'(0)}{|\vec{r}\,'(0)|} = \frac{\langle 1,0,1\rangle}{\sqrt{2}} = \langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\rangle$.

(2) We have

$$\vec{r}\,'(t) = \langle -e^{-t}, 2t + 2, e^t\rangle, \quad \vec{r}\,'(0) = \langle -1, 2, 1\rangle,$$

so $\vec{T}(0) = \frac{\vec{r}\,'(0)}{|\vec{r}\,'(0)|} = \frac{\langle -1,2,1\rangle}{\sqrt{1+4+1}} = \langle -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\rangle$.

(3) We have

$$\vec{r}\,'(t) = \langle \frac{1}{t+1}, -\sin(t), -\frac{2t}{(t^2-3)^2} \rangle, \quad \vec{r}\,'(0) = \langle 1, 0, 0 \rangle.$$

So, $\vec{T}(0) = \langle 1, 0, 0 \rangle$.

□

Exercise 5. Find the unit tangent vector $\vec{T}(t)$ at the given point.

(1) $\vec{r}(t) = \langle \frac{1}{t+1}, \frac{1}{t^2+1}, \sin^2 t \rangle, \quad (1, 1, 0)$
(2) $\vec{r}(t) = \langle \sin t, 5t, \cos t \rangle, \quad (0, 0, 1)$
(3) $\vec{r}(t) = \langle (t+1)^{2/3}, e^{t-7}, \frac{7}{t} \rangle, \quad (4, 1, 1)$

Solution.

(1) Note that $(1, 1, 0)$ corresponds to $t = 0$. We have

$$\vec{r}\,'(t) = \langle -\frac{1}{(t+1)^2}, -\frac{2t}{(t^2+1)^2}, 2\sin t\cos t \rangle, \quad \vec{r}\,'(0) = \langle -1, 0, 0 \rangle.$$

So $\vec{T}(0) = \langle -1, 0, 0 \rangle$.

(2) Note that $(0, 0, 1)$ corresponds to $t = 0$. We have

$$\vec{r}\,'(t) = \langle \cos t, 5, -\sin t \rangle, \quad \vec{r}\,'(0) = \langle 1, 5, 0 \rangle,$$

so $\vec{T}(0) = \frac{\langle 1,5,0 \rangle}{\sqrt{1+25}} = \langle \frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}, 0 \rangle$.

(3) Note that $(4, 1, 1)$ corresponds to $t = 7$. We have

$$\vec{r}\,'(t) = \langle \frac{2}{3}(t+1)^{-1/3}, e^{t-7}, -\frac{7}{t^2} \rangle, \quad \vec{r}\,'(7) = \langle \frac{1}{3}, 1, -\frac{1}{7} \rangle.$$

So,

$$\vec{T}(7) = \frac{\langle \frac{1}{3}, 1, -\frac{1}{7} \rangle}{\sqrt{\frac{1}{9} + 1 + \frac{1}{49}}} = \frac{\langle \frac{1}{3}, 1, -\frac{1}{7} \rangle}{\sqrt{\frac{499}{441}}} = \langle \frac{7}{\sqrt{499}}, \frac{21}{\sqrt{499}}, -\frac{3}{\sqrt{499}} \rangle.$$

□

Exercise 6. Find a vector equation for the tangent line to the curve

$$x = \sqrt{t^2+3}, \quad y = \ln(t^2+3), \quad z = t,$$

at $(2, \ln 4, 1)$.

Solution. Note that $(2, \ln 4, 1)$ corresponds to $t = 1$. Let $\vec{r}(t) = \langle \sqrt{t^2+3}, \ln(t^2+3), t \rangle$. Then

$$\vec{r}\,'(t) = \langle \frac{t}{\sqrt{t^2+3}}, \frac{2t}{t^2+3}, 1 \rangle,$$

so

$$\vec{r}\,'(1) = \langle \frac{1}{2}, \frac{1}{2}, 1 \rangle.$$

So the equation of the tangent line is

$$\langle 2, \ln 4, 1 \rangle + s\langle \frac{1}{2}, \frac{1}{2}, 1 \rangle.$$

□

Exercise 7. Evaluate the integral.

(1) $\int_1^3 (t\vec{i} - 2t\vec{j} - (4t^3+3)\vec{k})dt$.
(2) $\int \left(\frac{1}{1+t}\vec{i} + t^{2/3}\vec{j} + \sin(t)\vec{k}\right) dt$.
(3) $\int_0^{\pi/4} (\cos(2t)\vec{i} + e^t\vec{j} + \sin(t)\vec{k})dt$.

Solution.

(1)

$$\int_1^3 (t\vec{i} - 2t\vec{j} - (4t^3+3)\vec{k})dt = \int_1^3 tdt\vec{i} - \int_1^3 2tdt\vec{j} - \int_1^3 (4t^3+3)dt\vec{k}$$

$$= \frac{t^2}{2}|_{t=1}^{t=3}\vec{i} - t^2|_{t=1}^{t=3}\vec{j} - (t^4+3t)|_{t=1}^{t=3}\vec{k} = \left(\frac{9}{2} - \frac{1}{2}\right)\vec{i} - (9-1)\vec{j} - (3^4 + 9 - (1+3))\vec{k}$$

$$= 4\vec{i} - 8\vec{j} - 86\vec{k}.$$

(2)

$$\int \left(\frac{1}{1+t}\vec{i} + t^{2/3}\vec{j} + \sin(t)\vec{k}\right) dt = \ln(1+t)\vec{i} + \frac{3}{5}t^{5/3}\vec{j} - \cos(t)\vec{k} + \vec{C},$$

where $\vec{C}$ is a constant vector.

(3)

$$\int_0^{\pi/4} (\cos(2t)\vec{I} + e^t\vec{j} + \sin(t)\vec{k})dt = \int_0^{\pi/4} \cos(2t)dt\vec{i} + \int_0^{\pi/4} e^t dt\vec{j} + \int_0^{\pi/4} \sin(t)dt\vec{k}$$

$$= \frac{\sin(2t)}{2}|_{t=0}^{t=\pi/4}\vec{i} + e^t|_{t=0}^{t=\pi/4}\vec{j} - \cos(t)|_{t=0}^{t=\pi/4}\vec{k} = \frac{1}{2}\vec{i} + (e^{\pi/4} - 1)\vec{j} - (\frac{\sqrt{2}}{2} - 1)\vec{k}.$$

□

Exercise 8. Find $\vec{r}(t)$.

(1) $\vec{r}\,'(t) = 2t\vec{i} + e^t\vec{j} + \sqrt{t}\vec{k}, \quad \vec{r}(1) = \vec{i} + \vec{j} - \vec{k}, \quad \vec{r}(1) = \vec{i} + \vec{j} - \vec{k}$.
(2) $\vec{r}\,'(t) = \frac{1}{t+2}\vec{i} + (t+1)^2\vec{j} + e^{-2t}\vec{k}, \quad \vec{r}(0) = 3\vec{i} - \vec{k}$.
(3) $\vec{r}\,'(t) = \langle t + e^{-t}, 3t^2 - 2, \sin(t) - e^t\rangle, \quad \vec{r}(0) = \langle 3, -2, 1\rangle$.

Solution.

(1) We have $\vec{r}(t) = \int(2t\vec{i} + e^t\vec{j} + \sqrt{t}\vec{k})dt = t^2\vec{i} + e^t\vec{j} + \frac{2}{3}t^{3/2}\vec{k} + \vec{C}$, for a constant vector $\vec{C}$. We know $\vec{C}$ by putting $t = 1$,

$$\vec{i} + \vec{j} - \vec{k} = \vec{r}(1) = \vec{i} + e\vec{j} + \frac{2}{3}\vec{k} + \vec{C},$$

or

$$\vec{C} = (1-e)\vec{j} - \frac{5}{3}\vec{k}.$$

So

$$\vec{r}(t) = t^2\vec{i} + (e^t + 1 - e)\vec{j} + (\frac{2}{3}t^{3/2} - \frac{5}{3})\vec{k}.$$

(2) We have $\vec{r}(t) = \int(\frac{1}{t+2}\vec{i} + (t+1)^2\vec{j} + e^{-2t}\vec{k})dt = \ln(t+2)\vec{i} + \frac{1}{3}(t+1)^3\vec{j} - \frac{1}{2}e^{-2t}\vec{k} + \vec{C}$, for a constant vector. We know $\vec{C}$ by putting $t = 0$,

$$3\vec{i} - \vec{k} = \vec{r}(0) = \ln(2)\vec{i} + \frac{1}{3}\vec{j} - \frac{1}{2}\vec{k} + \vec{C},$$

or

$$\vec{C} = (3 - \ln 2)\vec{i} - \frac{1}{3}\vec{j} - \frac{1}{2}\vec{k}.$$

So

$$\vec{r}(t) = (\ln(t+2) + 3 - \ln 2)\vec{i} + \frac{(t+1)^3 - 1}{3}\vec{j} - \frac{e^{-2t} + 1}{2}\vec{k}.$$

(3) We have $\vec{r}(t) = \int \langle t + e^{-t}, 3t^2 - 2, \sin(t) - e^t \rangle dt = \langle \frac{t^2}{2} - e^{-t}, t^3 - 2t, -\cos(t) - e^t \rangle + \vec{C}$, for a constant vector $\vec{C}$. We know $\vec{C}$ by putting $t = 0$,

$$\langle 3, -2, 1 \rangle = \vec{r}(0) = \langle -1, 0, -2 \rangle + \vec{C},$$

or

$$\vec{C} = \langle 4, -2, 3 \rangle.$$

So

$$\vec{r}(t) = \langle \frac{t^2}{2} - e^{-t} + 4, t^3 - 2t - 2, -\cos(t) - e^t + 3 \rangle.$$

□

Exercise 9. If $\vec{r}(t)$ is some curve on the sphere $x^2 + y^2 + z^2 = 1$ (namely if $|\vec{r}(t)| = 1$ regardless of t), then show that $\vec{r}\,'(t)$ is always orthogonal to $\vec{r}(t)$.

Solution. Since $|\vec{r}(t)| = 1$, $\vec{r}(t) \cdot \vec{r}(t) = 1$. Taking $\frac{d}{dt}$ on both sides, we get

$$\frac{d}{dt}\left(\vec{r}(t) \cdot \vec{r}(t)\right) = 0.$$

By the product rule, we have

$$\vec{r}\,'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}\,'(t) = 0,$$

so $2\vec{r}(t) \cdot \vec{r}\,'(t) = 0$, or $\vec{r}(t) \cdot \vec{r}\,'(t) = 0$. So, $\vec{r}\,'(t)$ is orthogonal to $\vec{r}(t)$. □

9. Calculus for curves and motions

Exercise 1. Find the arclength of the curve segment.

(1) $\vec{r}(t) = \langle t, 3\cos t, 3\sin t \rangle$, $-5 \leq t \leq 5$.
(2) $\vec{r}(t) = \langle t, t^2, \frac{2t^3}{3} \rangle$, $0 \leq t \leq 1$.
(3) $\vec{r}(t) = \langle t^2, 9t, 4t^{3/2} \rangle$, $1 \leq t \leq 4$.

Solution.

(1) We have $\vec{r}\,'(t) = \langle 1, -3\sin t, 3\cos t \rangle$, so $|\vec{r}\,'(t)| = \sqrt{1^2 + 9\sin^2 t + 9\cos^2 t} = \sqrt{10}$. So the arclength is

$$\int_{-5}^{5} |\vec{r}\,'(t)| dt = \int_{-5}^{5} \sqrt{10} dt = 10\sqrt{10}.$$

(2) We have $\vec{r}\,'(t) = \langle 1, 2t, 2t^2 \rangle$, so $|\vec{r}\,'(t)| = \sqrt{1 + 4t^2 + 4t^4} = 2t^2 + 1$. So the arclength is

$$\int_0^1 |\vec{r}\,'(t)| dt = \int_0^1 (2t^2 + 1) dt = \frac{2t^3}{3} + t\Big|_{t=0}^{t=1} = \frac{2}{3} + 1 = \frac{5}{3}.$$

(3) We have $\vec{r}\,'(t) = \langle 2t, 9, 6t^{1/2} \rangle$, so $|\vec{r}\,'(t)| = \sqrt{4t^2 + 81 + 36t} = 2t + 9$. So the arclength is

$$\int_1^4 |\vec{r}\,'(t)| dt = \int_1^4 (2t + 9) dt = t^2 + 9t\Big|_{t=1}^{t=4} = (16 + 36) - (1 + 9) = 42.$$

□

Exercise 2. Find the arclength parametrization.

(1) $\vec{r}(t) = \langle \cos(t^3), \sin(t^3)\rangle$, starting from $t = 0$, to the direction of increasing t.
(2) $\vec{r}(t) = \langle \cos(t^4), \sin(t^4)\rangle$, starting from $t = 0$, to the direction of decreasing t.
(3) $\vec{r}(t) = \langle 5 - t, 4t - 3, 3t\rangle$, starting from $(5, -3, 0)$, to the direction of increasing t.
(4) $\vec{r}(t) = \langle e^t \sin t, e^t \cos t, \sqrt{2}e^t\rangle$, starting from $(0, -e^\pi, \sqrt{2}e^\pi)$, to the direction of decreasing t.
(5) $\vec{r}(t) = \langle e^t \sin t, e^t \cos t, \sqrt{2}e^t\rangle$, starting from $(0, -e^\pi, \sqrt{2}e^\pi)$, to the direction of increasing t.

Solution.

(1) We have $\vec{r}\,'(t) = \langle -3t^2 \sin(t^3), 3t^2 \cos(t^3)\rangle$, so $|\vec{r}\,'(t)| = 3t^2$. Since we are interested in the parametrization to the positive direction, we want to compute the arclength from time 0 to time t, denoted $\ell(t)$. This would be given as
$$\ell(t) = \int_0^t |\vec{r}\,'(s)|ds = \int_0^t 3s^2 ds = t^3.$$
So, the parametrization with respect to arclength ℓ from $t = 0$ to positive direction would be to replace t^3 by ℓ. This is just $\vec{r}(\ell) = \langle \cos(\ell), \sin(\ell)\rangle$.

As a sanity check, this should satisfy $|\vec{r}\,'(\ell)| = 1$ and that the parametric curve $\vec{r}(\ell)$ for $\ell \geq 0$ would correspond to the parametric curve $\vec{r}(t)$ for $t \geq 0$. It is indeed the case that $|\vec{r}\,'(\ell)| = 1$ because we did that calculation in class. Also $\ell \geq 0$ corresponds to $t \geq 0$ because the argument in $\cos$ and $\sin$ increases in both cases.

(2) We have $\vec{r}\,'(t) = \langle -4t^3 \sin(t^4), 4t^3 \cos(t^4)\rangle$, so $|\vec{r}\,'(t)| = 4|t|^3$. Since we are interested in the parametrization to the positive direction, we want to compute the arclength from time t to time 0, denoted $\ell(t)$. This would be given as
$$\ell(t) = \int_0^t |\vec{r}\,'(s)|ds = \int_t^0 4|s|^3 ds = \int_t^0 -4s^3 ds = -s^4\Big|_{s=t}^{s=0} = t^4.$$
So, the parametrization with respect to arclength ℓ from $t = 0$ to negative direction would be to replace t^4 by ℓ. This is just $\vec{r}(\ell) = \langle \cos(\ell), \sin(\ell)\rangle$.

As a sanity check, this should satisfy $|\vec{r}\,'(\ell)| = 1$ and that the parametric curve $\vec{r}(\ell)$ for $\ell \geq 0$ would correspond to the parametric curve $\vec{r}(t)$ for $t \leq 0$. It is indeed the case that $|\vec{r}\,'(\ell)| = 1$ because we did that calculation in class. Also $\ell \geq 0$ corresponds to $t \leq 0$ because the argument in $\cos$ and $\sin$ increases in both cases.

(3) We have $\vec{r}\,'(t) = \langle -1, 4, 3\rangle$, so $|\vec{r}\,'(t)| = \sqrt{1 + 16 + 9} = \sqrt{26}$. Note that $(5, -3, 0)$ corresponds to $t = 0$. Since we are interested in the parametrization to the positive direction, we want to compute the arclength from time 0 to time t, denoted $\ell(t)$. This would be given as
$$\ell(t) = \int_0^t |\vec{r}\,'(s)|ds = \int_0^t \sqrt{26}ds = \sqrt{26}t.$$
So, the parametrization with respect to arclength ℓ from $t = 0$ to positive direction would be to replace $\sqrt{26}t$ by ℓ, or $t = \frac{\ell}{\sqrt{26}}$. Thus, the parametrization we are looking for is
$$\vec{r}(\ell) = \langle 5 - \frac{\ell}{\sqrt{26}}, \frac{4\ell}{\sqrt{26}} - 3, \frac{3\ell}{\sqrt{26}}\rangle.$$

As a sanity check, this should satisfy $|\vec{r}\,'(\ell)| = 1$ and that the parametric curve $\vec{r}(\ell)$ for $\ell \geq 0$ would correspond to the parametric curve $\vec{r}(t)$ for $t \geq 0$. Indeed, as ℓ increases, x decreases, y increases, and z increases, which is the same when t increases. Also, $\vec{r}\,'(\ell) = \langle -\frac{1}{\sqrt{26}}, \frac{4}{\sqrt{26}}, \frac{3}{\sqrt{26}} \rangle$, so $|\vec{r}\,'(\ell)| = \sqrt{\frac{1}{26} + \frac{16}{26} + \frac{9}{26}} = \sqrt{\frac{26}{26}} = 1$.

(4) We have $\vec{r}\,'(t) = \langle e^t \cos t + e^t \sin t, -e^t \sin t + e^t \cos t, \sqrt{2}e^t \rangle$, so

$$|\vec{r}\,'(t)| = \sqrt{(e^t \cos t + e^t \sin t)^2 + (-e^t \sin t + e^t \cos t)^2 + (\sqrt{2}e^t)^2}$$
$$= e^t \sqrt{(\cos t + \sin t)^2 + (\cos t - \sin t)^2 + 2}$$
$$= e^t \sqrt{\cos^2 t + 2\cos t \sin t + \sin^2 t + \cos^2 t - 2\cos t \sin t + \sin^2 t + 2}$$
$$= e^t \sqrt{2\cos^2 t + 2\sin^2 t + 2} = e^t \sqrt{4} = 2e^t.$$

Note that $(0, -e^\pi, \sqrt{2}e^\pi)$ corresponds to $t = \pi$. Since we are interested in the parametrization to the negative direction, we want to compute the arclength from time t to time π, denoted $\ell(t)$. This would be given as

$$\ell(t) = \int_t^\pi |\vec{r}\,'(s)| ds = \int_t^\pi 2e^s ds = 2e^s \Big|_{s=t}^{s=\pi} = 2e^\pi - 2e^t.$$

Thus, the parametrization with respect to arclength ℓ from $t = \pi$ to negative direction would be to replace $2e^\pi - 2e^t$ by ℓ, or $2e^\pi - 2e^t = \ell$, or $2e^\pi - \ell = 2e^t$, or $e^t = \frac{2e^\pi - \ell}{2}$, or $t = \ln\left(\frac{2e^\pi - \ell}{2}\right)$. Thus

$$\vec{r}(\ell) = \langle e^{\ln\left(\frac{2e^\pi-\ell}{2}\right)} \sin\left(\ln\left(\frac{2e^\pi - \ell}{2}\right)\right), e^{\ln\left(\frac{2e^\pi-\ell}{2}\right)} \cos\left(\ln\left(\frac{2e^\pi - \ell}{2}\right)\right), \sqrt{2}e^{\ln\left(\frac{2e^\pi-\ell}{2}\right)} \rangle$$
$$= \langle \frac{2e^\pi - \ell}{2} \sin\left(\ln\left(\frac{2e^\pi - \ell}{2}\right)\right), \frac{2e^\pi - \ell}{2} \cos\left(\ln\left(\frac{2e^\pi - \ell}{2}\right)\right), \sqrt{2}\frac{2e^\pi - \ell}{2} \rangle$$

As a sanity check, this should satisfy $|\vec{r}\,'(\ell)| = 1$ and that the parametric curve $\vec{r}(\ell)$ for $\ell \geq 0$ would correspond to the parametric curve $\vec{r}(t)$ for $t \leq 0$. Indeed, as ℓ increases, z decreases, which is the case when t decreases. Also,

$$\vec{r}\,'(\ell) = \langle -\frac{1}{2} \sin\left(\ln\left(\frac{2e^\pi - \ell}{2}\right)\right) + \frac{2e^\pi - \ell}{2} \cdot \left(-\frac{1}{2} \frac{\cos\left(\ln\left(\frac{2e^\pi-\ell}{2}\right)\right)}{\frac{2e^\pi-\ell}{2}}\right),$$
$$-\frac{1}{2} \cos\left(\ln\left(\frac{2e^\pi - \ell}{2}\right)\right) + \frac{2e^\pi - \ell}{2} \cdot \left(-\frac{1}{2} \frac{-\sin\left(\ln\left(\frac{2e^\pi-\ell}{2}\right)\right)}{\frac{2e^\pi-\ell}{2}}\right), -\frac{\sqrt{2}}{2} \rangle$$
$$= \langle -\frac{1}{2} \sin\left(\ln\left(\frac{2e^\pi - \ell}{2}\right)\right) - \frac{1}{2} \cos\left(\ln\left(\frac{2e^\pi - \ell}{2}\right)\right),$$
$$-\frac{1}{2} \cos\left(\ln\left(\frac{2e^\pi - \ell}{2}\right)\right) + \frac{1}{2} \sin\left(\ln\left(\frac{2e^\pi - \ell}{2}\right)\right), -\frac{\sqrt{2}}{2} \rangle$$

So

$$|\vec{r}\,'(\ell)|^2 = \left(-\frac{1}{2} \sin\left(\ln\left(\frac{2e^\pi - \ell}{2}\right)\right) - \frac{1}{2} \cos\left(\ln\left(\frac{2e^\pi - \ell}{2}\right)\right)\right)^2$$
$$+ \left(-\frac{1}{2} \cos\left(\ln\left(\frac{2e^\pi - \ell}{2}\right)\right) + \frac{1}{2} \sin\left(\ln\left(\frac{2e^\pi - \ell}{2}\right)\right)\right)^2 + \frac{1}{2}$$

$$= \frac{1}{4}\sin^2\left(\ln\left(\frac{2e^\pi - \ell}{2}\right)\right) + \frac{1}{2}\sin\left(\ln\left(\frac{2e^\pi - \ell}{2}\right)\right)\cos\left(\ln\left(\frac{2e^\pi - \ell}{2}\right)\right) + \frac{1}{4}\cos^2\left(\ln\left(\frac{2e^\pi - \ell}{2}\right)\right)$$

$$+ \frac{1}{4}\cos^2\left(\ln\left(\frac{2e^\pi - \ell}{2}\right)\right) - \frac{1}{2}\cos\left(\ln\left(\frac{2e^\pi - \ell}{2}\right)\right)\sin\left(\ln\left(\frac{2e^\pi - \ell}{2}\right)\right) + \frac{1}{4}\sin^2\left(\ln\left(\frac{2e^\pi - \ell}{2}\right)\right) + \frac{1}{2}$$

$$= \frac{1}{2}\sin^2\left(\ln\left(\frac{2e^\pi - \ell}{2}\right)\right) + \frac{1}{2}\cos^2\left(\ln\left(\frac{2e^\pi - \ell}{2}\right)\right) + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1$$

which is what we want.

(5) Since we are interested in the parametrization to the negative direction, we want to compute the arclength from time π to time t, denoted $\ell(t)$. This would be given as

$$\ell(t) = \int_\pi^t |\vec{r}\,'(s)|ds = \int_\pi^t 2e^s ds = 2e^s\Big|_{s=\pi}^{s=t} = 2e^t - 2e^\pi.$$

Thus, the parametrization with respect to arclength ℓ from $t = \pi$ to positive direction would be to replace $2e^t - 2e^\pi$ by ℓ, or $2e^t - 2e^\pi = \ell$, or $e^t = \frac{\ell + 2e^\pi}{2}$, or $t = \ln\left(\frac{\ell+2e^\pi}{2}\right)$. Thus

$$\vec{r}(\ell) = \langle e^{\ln\left(\frac{\ell+2e^\pi}{2}\right)}\sin\left(\ln\left(\frac{\ell + 2e^\pi}{2}\right)\right), e^{\ln\left(\frac{\ell+2e^\pi}{2}\right)}\cos\left(\ln\left(\frac{\ell + 2e^\pi}{2}\right)\right), \sqrt{2}e^{\ln\left(\frac{\ell+2e^\pi}{2}\right)}\rangle$$

$$= \langle\frac{\ell + 2e^\pi}{2}\sin\left(\ln\left(\frac{\ell + 2e^\pi}{2}\right)\right), \frac{\ell + 2e^\pi}{2}\cos\left(\ln\left(\frac{\ell + 2e^\pi}{2}\right)\right), \sqrt{2}\frac{\ell + 2e^\pi}{2}\rangle$$

As a sanity check, this should satisfy $|\vec{r}\,'(\ell)| = 1$ and that the parametric curve $\vec{r}(\ell)$ for $\ell \geq 0$ would correspond to the parametric curve $\vec{r}(t)$ for $t \geq 0$. Indeed, as ℓ increases, z increases, which is also the case when t increases. Also,

$$\vec{r}\,'(\ell) = \langle\frac{1}{2}\sin\left(\ln\left(\frac{\ell + 2e^\pi}{2}\right)\right) + \frac{\ell + 2e^\pi}{2}\cdot\left(\frac{1}{2}\frac{\cos\left(\ln\left(\frac{\ell+2e^\pi}{2}\right)\right)}{\frac{\ell+2e^\pi}{2}}\right),$$

$$\frac{1}{2}\cos\left(\ln\left(\frac{\ell + 2e^\pi}{2}\right)\right) - \frac{\ell + 2e^\pi}{2}\cdot\left(\frac{1}{2}\frac{\sin\left(\ln\left(\frac{\ell+2e^\pi}{2}\right)\right)}{\frac{\ell+2e^\pi}{2}}\right), \frac{\sqrt{2}}{2}\rangle$$

$$= \frac{1}{2}\langle\sin\left(\ln\left(\frac{\ell + 2e^\pi}{2}\right)\right) + \cos\left(\ln\left(\frac{\ell + 2e^\pi}{2}\right)\right),$$

$$\cos\left(\ln\left(\frac{\ell + 2e^\pi}{2}\right)\right) - \sin\left(\ln\left(\frac{\ell + 2e^\pi}{2}\right)\right), \sqrt{2}\rangle$$

So

$$|\vec{r}\,'(\ell)|^2 = \frac{1}{4}(\left(\sin\left(\ln\left(\frac{\ell + 2e^\pi}{2}\right)\right) + \cos\left(\ln\left(\frac{\ell + 2e^\pi}{2}\right)\right)\right)^2$$

$$+ \left(\cos\left(\ln\left(\frac{\ell + 2e^\pi}{2}\right)\right) - \sin\left(\ln\left(\frac{\ell + 2e^\pi}{2}\right)\right)\right)^2 + 2)$$

$$= \frac{1}{4}(\sin^2\left(\ln\left(\frac{\ell + 2e^\pi}{2}\right)\right) + 2\sin\left(\ln\left(\frac{\ell + 2e^\pi}{2}\right)\right)\cos\left(\ln\left(\frac{\ell + 2e^\pi}{2}\right)\right) + \cos^2\left(\ln\left(\frac{\ell + 2e^\pi}{2}\right)\right)$$

$$+ \cos^2\left(\ln\left(\frac{\ell + 2e^\pi}{2}\right)\right) - 2\cos\left(\ln\left(\frac{\ell + 2e^\pi}{2}\right)\right)\sin\left(\ln\left(\frac{\ell + 2e^\pi}{2}\right)\right) + \sin^2\left(\ln\left(\frac{\ell + 2e^\pi}{2}\right)\right) + 2)$$

$$= \frac{1}{4}\left(2\sin^2\left(\ln\left(\frac{\ell + 2e^\pi}{2}\right)\right) + 2\cos^2\left(\ln\left(\frac{\ell + 2e^\pi}{2}\right)\right) + 2\right) = \frac{1}{4}(2 + 2) = 1.$$

□

Exercise 3. Explain why the parametrization of $\vec{r}(t) = \langle \cos(t^2), \sin(t^2)\rangle$ with respect to arclength, starting from $t = -3$, to the positive direction, does not exist.

Solution. The arclength parametrization exists only if the parametric curve travels in a fixed direction. On the other hand, the given parametric curve travels clockwise from $t = -3$ to $t = 0$, and at $t = 0$ it turns around and starts travelling counterclockwise. □

Exercise 4. An object moves with position function $\vec{r}(t) = \langle t^2, e^t \sin t, e^t \cos t\rangle$. Find the velocity and acceleration $\vec{v}(t)$ and $\vec{a}(t)$.

Solution. We just use $\vec{v}(t) = \vec{r}\,'(t)$, and $\vec{a}(t) = \vec{r}\,''(t) = \vec{v}\,'(t)$. We have

$$\vec{v}(t) = \vec{r}\,'(t) = \langle 2t, e^t \cos t + e^t \sin t, -e^t \sin t + e^t \cos t\rangle.$$

$$\vec{a}(t) = \vec{r}\,''(t) = \vec{v}\,'(t) = \langle 2, (e^t \cos t - e^t \sin t) + (e^t \sin t + e^t \cos t), (-e^t \sin t - e^t \cos t) + (e^t \cos t - e^t \sin t)\rangle$$

$$= \langle 2, 2e^t \cos t, -2e^t \sin t\rangle.$$

□

10. Functions of several variables

Exercise 1. Find the values of the following functions.

(1) $f(2, 5)$ for $f(x, y) = x^2y^2 - \frac{x}{y}$.
(2) $f(\frac{\pi}{2}, \frac{\pi}{3})$ for $f(x, y) = x \sin y + y \sin x$.
(3) $f(0, \pi, 1)$ for $f(x, y, z) = x^2y^3z - xe^z + z \sin y$.

Solution.

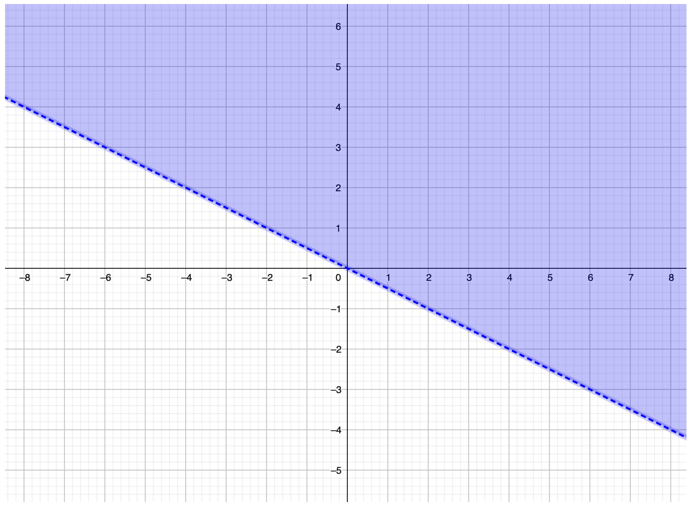
(1) $f(2, 5) = 2^2 \cdot 5^2 - \frac{2}{5} = 100 - \frac{2}{5} = \frac{498}{5}$.
(2) $f(\frac{\pi}{2}, \frac{\pi}{3}) = \frac{\pi}{2} \sin \frac{\pi}{3} + \frac{\pi}{3} \sin \frac{\pi}{2} = \frac{\pi}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\pi}{3} \cdot 1 = \frac{\pi\sqrt{3}}{4} + \frac{\pi}{3} = \frac{\pi(3\sqrt{3}+4)}{12}$.
(3) $f(0, \pi, 1) = 0^2 \cdot \pi^3 \cdot 1 - 0 \cdot e^0 + 1 \cdot \sin \pi = 0$.

□

Exercise 2. Find and sketch the domain of the following functions.

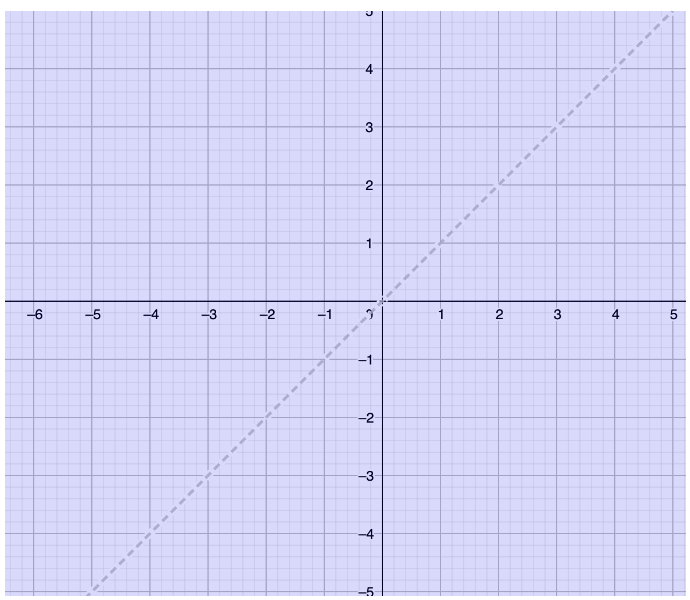
(1) $f(x, y) = \ln(x + 2y)$
(2) $f(x, y) = \frac{1}{x+y}$
(3) $f(x, y) = \sqrt{x^2 - y^2}$
(4) $f(x, y) = \frac{1}{\sqrt{x - \sin y}}$

Solution. (1) The domain is $x + 2y > 0$.



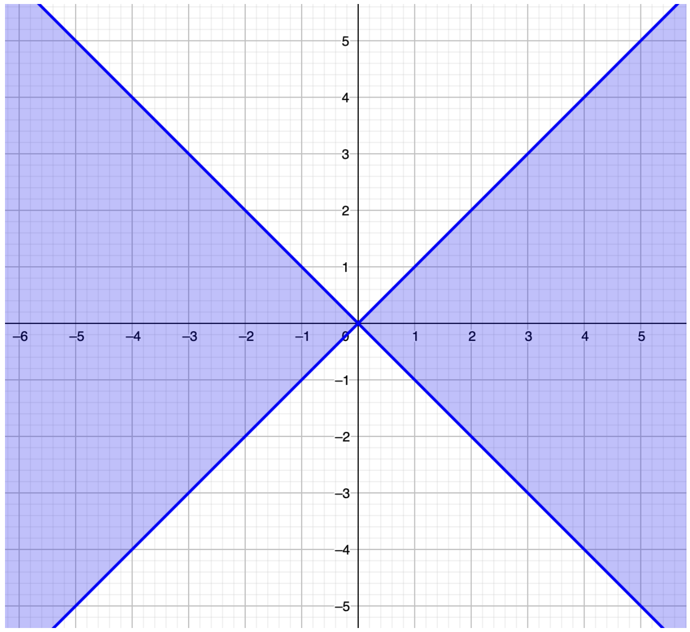
The dotted line means the boundary line is **not included**.

(2) The domain is $x + y \neq 0$, or $x \neq -y$.



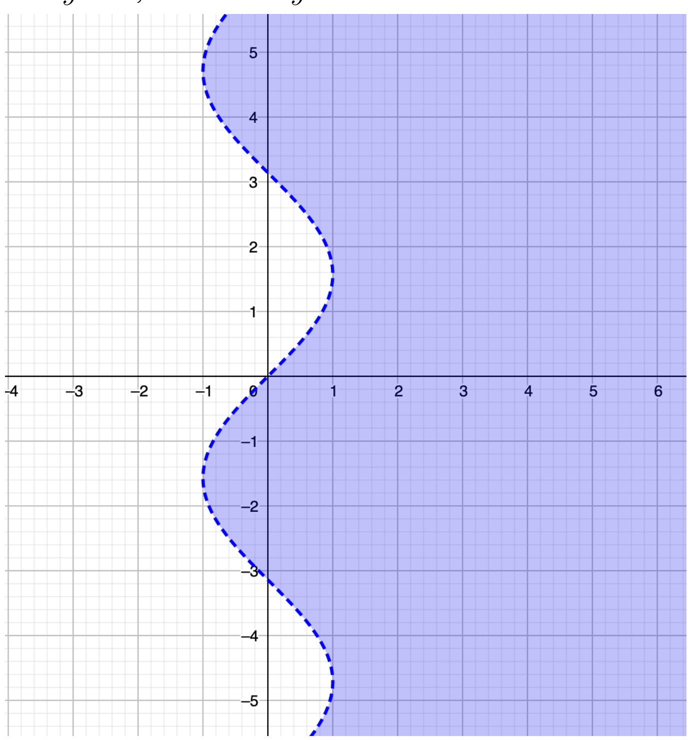
The dotted grey line is in particular **not included**.

(3) The domain is $x^2 - y^2 \geq 0$, or $x^2 \geq y^2$, or $|x| \geq |y|$.



The boundary lines are included.

(4) The domain is $x - \sin y > 0$, or $x > \sin y$.



The dotted boundary means the boundary curve is **not included**.

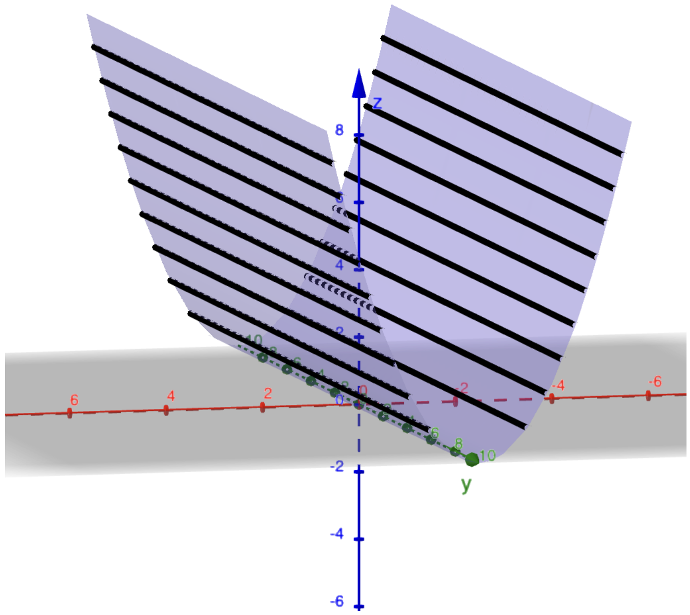
□

Exercise 3. Sketch the graph of the following functions. Draw some of their horizontal traces.

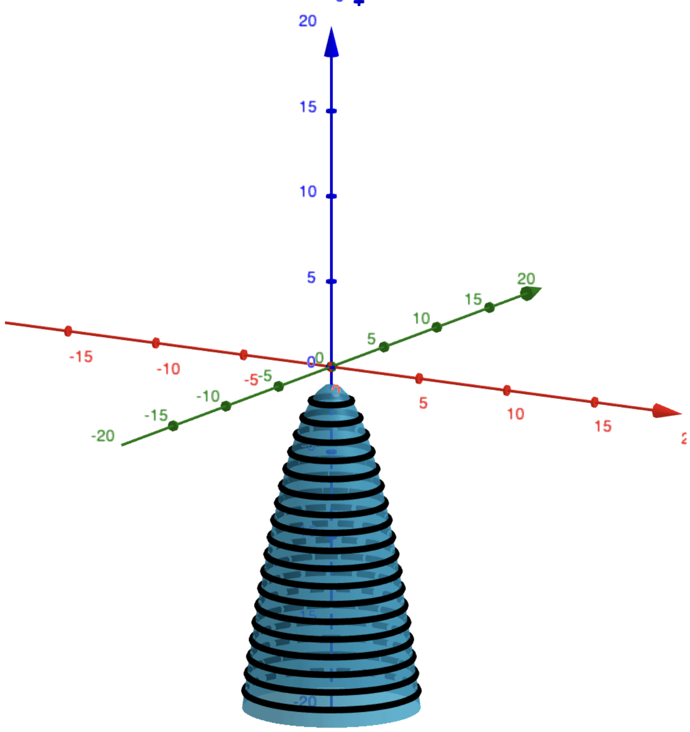
(1) $f(x, y) = x^2$
(2) $f(x, y) = -1 - x^2 - y^2$
(3) $f(x, y) = xy$

(4) The implicit equation $x^2 + y^2 + 4z^2 = 1$ (with z as an implicit function of x, y).

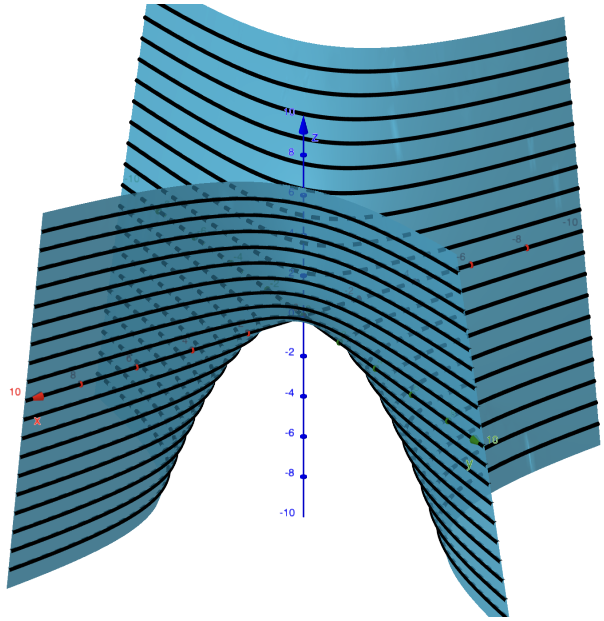
Solution.

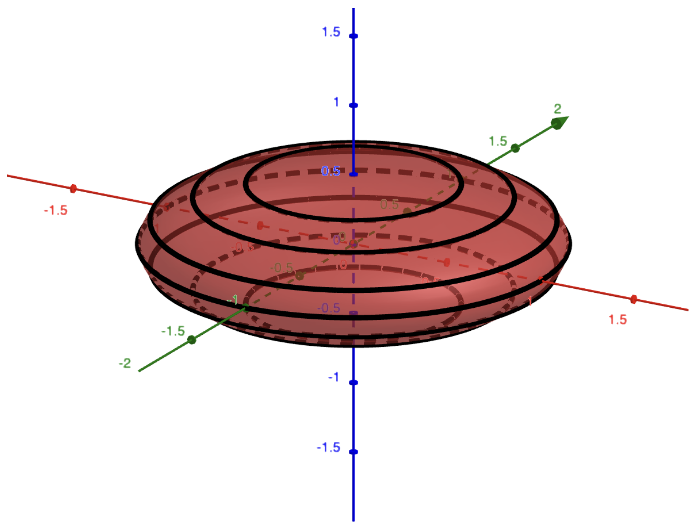


(1)



(2)

(3) 

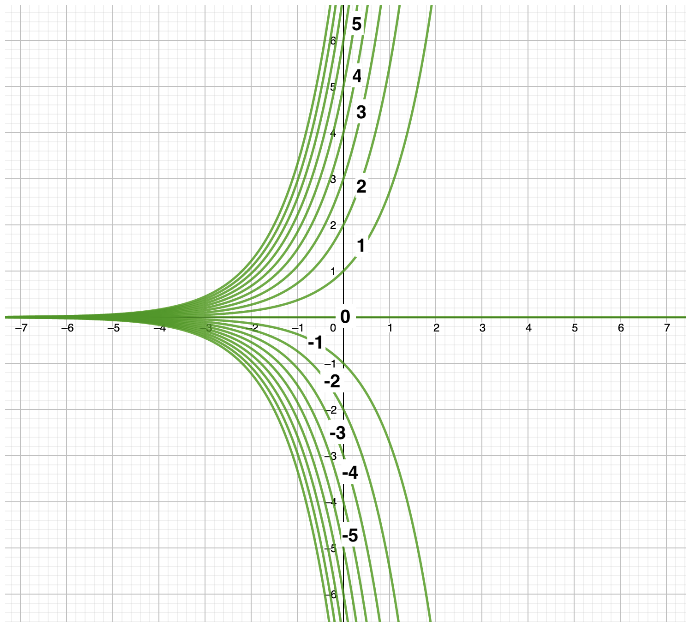
(4) 

□

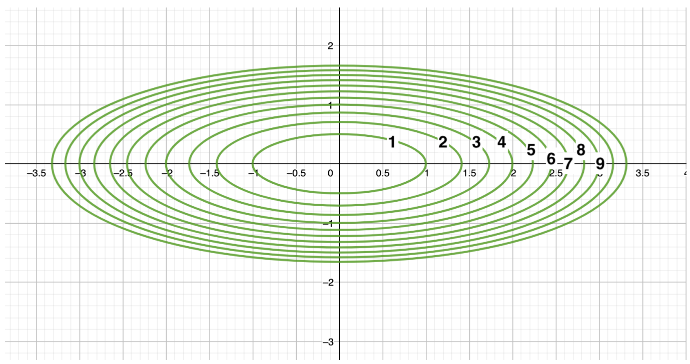
Exercise 4. Sketch the contour map of the following functions.

(1) $f(x, y) = ye^{-x}$
(2) $f(x, y) = x^2 + 4y^2$

Solution.



(1)



(2)

□

11. Limits and continuity in several variables

Exercise 1. Find the limit.

(1)

$$\lim_{(x,y)\to(2,1)} (x^3y - 3y^2)$$

(2)

$$\lim_{(x,y)\to(0,-10)} (x^4y^2 + 3xy - 2y + 4)$$

(3)

$$\lim_{(x,y)\to(2,2)} \frac{\sin(x-y)}{x-y}$$

(4)

$$\lim_{(x,y)\to(1,-1)} \sin\left(\frac{e^{(x+y)^2}-1}{x+y}\right)$$

(5)

$$\lim_{(x,y)\to(1,1)} \left(\frac{x^2y^3 - x^3y^2}{x^2 - y^2} \right)$$

(Hint: $x^2y^3 - x^3y^2 = x^2y^2(y-x)$, and $x^2 - y^2 = (x-y)(x+y)$)

(6)

$$\lim_{(x,y)\to(0,0)} xy \sin\left(\frac{1}{x^2+y^2} \right)$$

(7)

$$\lim_{(x,y)\to(0,0)} \left(\frac{xy^2 \sin(x)}{x^2+y^2} \right)$$

(8)

$$\lim_{(x,y)\to(0,0)} \left(1 + \frac{x^3 \sin(x)}{x^2+y^4} \right)$$

Solution.

(1) You plug $x = 2, y = 1$ to get

$$\lim_{(x,y)\to(2,1)} (x^3y - 3y^2) = 8 \cdot 1 - 3 \cdot 1^2 = 5.$$

(2) You plug $x = 0, y = -10$ to get

$$\lim_{(x,y)\to(0,-10)} (x^4y^2 + 3xy - 2y + 4) = 0^4 \cdot (-10)^2 + 3 \cdot 0 \cdot (-10) - 2 \cdot (-10) + 4 = 24.$$

(3) The function $f(x,y) = \frac{\sin(x-y)}{x-y}$ is the composition $f(x,y) = g(h(x,y))$ where $g(x) = \frac{\sin(x)}{x}$ and $h(x,y) = x - y$. Note that

$$\lim_{(x,y)\to(2,2)} h(x,y) = \lim_{(x,y)\to(2,2)} x - y = 0$$

and

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

so

$$\lim_{(x,y)\to(2,2)} \frac{\sin(x-y)}{x-y} = \lim_{(x,y)\to(2,2)} g(h(x,y)) = \lim_{h\to 0} g(h) = 1$$

(4) The function $f(x,y) = \sin\left(\frac{e^{(x+y)^2} - 1}{x+y} \right)$ is the composition $f(x,y) = g(h(i(x,y)))$ where $g(x) = \sin(x)$, $h(x) = \frac{e^{x^2}-1}{x}$ and $i(x,y) = x + y$. Note that

$$\lim_{(x,y)\to(1,-1)} i(x,y) = \lim_{(x,y)\to(1,-1)} x + y = 0$$

and

$$\lim_{x\to 0} h(x) = \lim_{x\to 0} \frac{e^{x^2} - 1}{x}$$

which is of form $\frac{0}{0}$, so by L'Hopital

$$\lim_{x\to 0} h(x) = \lim_{x\to 0} \frac{e^{x^2} - 1}{x} = \lim_{x\to 0} \frac{2xe^{x^2}}{1} = 0$$

and

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \sin(x) = 0.$$

So

$$\lim_{(x,y)\to(1,-1)} \sin\left(\frac{e^{(x+y)^2}-1}{x+y}\right) = \lim_{(x,y)\to(1,-1)} g(h(i(x,y))) = \lim_{i\to 0} g(h(i)) = \lim_{h\to 0} g(h) = 0.$$

(5) Note that by Hint

$$\frac{x^2y^3 - x^3y^2}{x^2-y^2} = \frac{x^2y^2(y-x)}{(x-y)(x+y)} = \frac{x^2y^2}{-(x+y)}$$

so

$$\lim_{(x,y)\to(1,1)} \left(\frac{x^2y^3 - x^3y^2}{x^2-y^2}\right) = \lim_{(x,y)\to(1,1)} \left(\frac{x^2y^2}{-(x+y)}\right) = -\frac{1}{2}$$

(6) Note that

$$\left|\sin\left(\frac{1}{x^2+y^2}\right)\right| \leq 1$$

so

$$\left|xy\sin\left(\frac{1}{x^2+y^2}\right)\right| \leq |xy|.$$

Since

$$\lim_{(x,y)\to(0,0)} xy = 0$$

by Squeeze Theorem,

$$\lim_{(x,y)\to(0,0)} xy\sin\left(\frac{1}{x^2+y^2}\right) = 0$$

(7) Note that $|\sin(x)| \leq 1$. Note also that $\left|\frac{xy^2}{x^2+y^2}\right| \leq |x|$, because $\frac{y^2}{x^2+y^2} \leq 1$. Since $\lim_{(x,y)\to(0,0)} x = 0$, by Squeeze Theorem

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = 0,$$

and again by Squeeze Theorem

$$\lim_{(x,y)\to(0,0)} \frac{xy^2\sin(x)}{x^2+y^2} = 0.$$

(8) Note that

$$\lim_{(x,y)\to(0,0)} \left(1 + \frac{x^3\sin(x)}{x^2+y^4}\right) = 1 + \lim_{(x,y)\to(0,0)} \left(\frac{x^3\sin(x)}{x^2+y^4}\right)$$

so you are left with finding what

$$\lim_{(x,y)\to(0,0)} \left(\frac{x^3\sin(x)}{x^2+y^4}\right)$$

is. Note that since $|\sin(x)| \leq 1$, we have $\left|\frac{x^3\sin(x)}{x^2+y^4}\right| \leq \left|\frac{x^3}{x^2+y^4}\right| \leq \frac{|x|^3}{x^2} = |x|$. Since $\lim_{(x,y)\to(0,0)} x = 0$, we have by Squeeze Theorem

$$\lim_{(x,y)\to(0,0)} \left(\frac{x^3\sin(x)}{x^2+y^4}\right) = 0.$$

So

$$\lim_{(x,y)\to(0,0)} \left(1+\frac{x^3\sin(x)}{x^2+y^4}\right) = 1$$

□

Exercise 2. Show that the limit does not exist.

(1)
$$\lim_{(x,y)\to(0,0)} \frac{x}{2x+y}$$

(2)
$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+3y^2}$$

(3)
$$\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{x^4+y^6}$$

(4)
$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^4-2x^2y^2-y^4}$$

(5)
$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^5+3x^2y^2-y^3}$$

(6)
$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy^2+yz^2+zx^2}{x^3+y^3+z^3}$$

(7)
$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^3y^2z}{x^6+x^3y^3+y^6+z^6}$$

(8)
$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^2y+yz^2}{x^5+x^3y+x^2z^2+xy^2z}$$

Solution.

(1) Use the path $x = 0$ and y approaching 0, we get $\frac{x}{2x+y} = 0$. On the other hand, if we use the path $x = y$ approaching 0, we get $\frac{x}{2x+y} = \frac{x}{3x} = \frac{1}{3}$, so the limit along the second path is $\frac{1}{3}$. So the limit does not exist.

(2) Use the path $x = 0$ and y approaching 0, we get $\frac{xy}{x^2+3y^2} = 0$. On the other hand, if we use the path $x = y$ approaching 0, we get $\frac{xy}{x^2+3y^2} = \frac{x^2}{4x^2} = \frac{1}{4}$, so the limit along the second path is $\frac{1}{4}$. So the limit does not exist.

(3) Use the path $x = 0$ and y approaching 0, we get $\frac{x^2y^3}{x^4+y^6} = 0$. On the other hand, if we use the path $y = x^{2/3}$ approaching 0, we get $\frac{x^2y^3}{x^4+y^6} = \frac{x^2x^2}{x^4+x^4} = \frac{1}{2}$. So the limit along the second path is $\frac{1}{2}$. So the limit does not exist.

(4) Use the path $x = 0$ and y approaching 0, we get $\frac{x^3y}{x^4-2x^2y^2-y^4} = 0$. On the other hand, if we use the path $y = x$ approaching 0, we get $\frac{x^3y}{x^4-2x^2y^2-y^4} = \frac{x^4}{x^4-2x^4-x^4} = -\frac{1}{2}$. So the limit along the second path is $-\frac{1}{2}$. So the limit does not exist.

(5) Use the path $x = 0$ and y approaching 0, we get $\frac{x^3y}{x^5+3x^2y^2-y^3} = 0$. On the other hand, if we use the path $y = x^2$ approaching 0, we get $\frac{x^3y}{x^5+3x^2y^2-y^3} = \frac{x^5}{x^5+3x^6-x^6} = \frac{x^5}{x^5+2x^6} = \frac{1}{1+2x}$. So the limit along the second path is $\lim_{x\to 0} \frac{1}{1+2x} = 1$. So the limit does not exist.

(6) Use the path $x = y = 0$ and z approaching 0, we get $\frac{xy^2+yz^2+zx^2}{x^3+y^3+z^3} = 0$. On the other hand, if we use the path $x = y = z$ approaching 0, we get $\frac{xy^2+yz^2+zx^2}{x^3+y^3+z^3} = \frac{x^3+x^3+x^3}{x^3+x^3+x^3} = 1$. So the limit along the second path is 1. So the limit does not exist.

(7) Use the path $x = y = 0$ and z approaching 0, we get $\frac{x^3y^2z}{x^6+x^3y^3+y^6+z^6} = 0$. On the other hand, if we use the path $x = y = z$ approaching 0, we get $\frac{x^3y^2z}{x^6+x^3y^3+y^6+z^6} = \frac{x^6}{4x^6} = \frac{1}{4}$. So the limit along the second path is $\frac{1}{4}$. So the limit does not exist.

(8) Use the path $x = z = 0$ and y approaching 0, we get $\frac{x^2y+yz^2}{x^5+x^3y+x^2z^2+xy^2z} = 0$. On the other hand, if we use the path $y = x^3, z = x^2$ approaching 0, we get $\frac{x^2y+yz^2}{x^5+x^3y+x^2z^2+xy^2z} = \frac{x^5+x^7}{x^5+x^6+x^6+x^9} = \frac{x^5+x^7}{x^5+2x^6+x^9} = \frac{1+x^2}{1+x+x^4}$. So the limit along the second path is $\lim_{x\to 0} \frac{1+x^2}{1+x+x^4} = 1$. So the limit does not exist.

□

Exercise 3. Find the limit, if it exists, or show that the limit does not exist.

(1)
$$\lim_{(x,y)\to(1,2)} \frac{2x-y}{4x^2-y^2}$$

(2)
$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^4+y^2+z^3}{x^4+2y^2+z}$$

(3)
$$\lim_{(x,y)\to(1,1)} \frac{y-x}{1-y+\ln x}$$

(4)
$$\lim_{(x,y)\to(0,0)} \frac{y^2\sin^2 x}{x^4+y^4}$$

Solution.

(1) Since $4x^2 - y^2 = (2x-y)(2x+y)$, $\frac{2x-y}{4x^2-y^2} = \frac{1}{2x+y}$. So
$$\lim_{(x,y)\to(1,2)} \frac{2x-y}{4x^2-y^2} = \lim_{(x,y)\to(1,2)} \frac{1}{2x+y} = \frac{1}{4}$$

(2) If you approach along $y = z = 0$, we get $\frac{x^4+y^2+z^3}{x^4+2y^2+z} = \frac{x^4}{x^4} = 1$, so the limit is 1. On the other hand, if you approach along $x = y = 0$, then we get $\frac{x^4+y^2+z^3}{x^4+2y^2+z} = \frac{z^3}{z} = z^2$, so the limit is 0. So the limit does not exist.

(3) If you approach along $y = x$, we get $\frac{y-x}{1-y+\ln x} = 0$. On the other hand, if you approach along $x = e^{2y-2}$, then $\frac{y-x}{1-y+\ln x} = \frac{y-e^{2y-2}}{1-y+2y-2} = \frac{y-e^{2y-2}}{y-1}$, so the limit is, by L'Hopital,

$$\lim_{y\to 1} \frac{y - e^{2y-2}}{y-1} = \lim_{y\to 1} \frac{1-2e^{y-2}}{1} = 1 - 2e^{-1},$$

which is different from 0. So the limit does not exist.

(4) If you approach along $y = 0$, we get $\frac{y^2 \sin^2 x}{x^4+y^4} = 0$. On the other hand, if you approach along $x = y$, you get $\frac{y^2 \sin^2 x}{x^4+y^4} = \frac{x^2 \sin^2 x}{2x^4} = \frac{\sin^2 x}{2x^2}$, so the limit is $\lim_{x\to 0} \frac{\sin^2 x}{2x^2} = \frac{1}{2}$. So the limit does not exist.

□

12. Partial derivatives

Exercise 1. Find the indicated partial derivative.

(1) $f(x, y) = x^2y - 3y^4$, f_x
(2) $f(x, y) = \frac{x+y}{x-2y}$, f_y
(3) $f(x, y, z) = xy^2e^{-xz}$, f_z
(4) $f(x, y, z) = \ln(x + 2y + 3z)$, f_y
(5) $f(x, y) = xe^{y/x}$, $f_x(1, 0)$
(6) $f(x, y) = x^4y^2 - x^3e^y$, f_{xy}
(7) $f(x, y) = \frac{y}{2x+3y}$, f_{xy}
(8) $f(x, y) = \sin(x^2 - y^2)$, f_{xx}
(9) $f(x, y, z) = e^{xyz^2}$, f_{xyz}
(10) $f(x, y) = \sin(4x - 3y)$, f_{xyx}
(11) $f(x, y, z) = e^{x+y} - e^{\sin(y-z)}$, f_{xyz}
(Hint: compute $(e^{x+y})_{xyz}$ and $(e^{\sin(y-z)})_{xyz}$ separately)

Solution.

(1) $f_x = 2xy$
(2) $f_y = \frac{1}{x-2y} - \frac{(x+y)\cdot(-2)}{(x-2y)^2} = \frac{1}{x-2y} + \frac{2x+2y}{(x-2y)^2} = \frac{3x}{(x-2y)^2}$
(3) $f_z = xy^2(-x)e^{-xz} = -x^2y^2e^{-xz}$
(4) $f_y = \frac{2}{x+2y+3z}$
(5) $f_x = e^{y/x} + xe^{y/x} \cdot \left(-\frac{y}{x^2}\right) = e^{y/x} - \frac{y}{x}e^{y/x}$, so $f_x(1, 0) = e^0 = 1$.
(6) $f_x = 4x^3y^2 - 3x^2e^y$, and $f_{xy} = 8x^3y - 3x^2e^y$
(7) $f_x = -\frac{2y}{(2x+3y)^2}$, and $f_{xy} = -\frac{2}{(2x+3y)^2} + \frac{2\cdot(2y)\cdot 3}{(2x+3y)^3} = \frac{-2(2x+3y)+12y}{(2x+3y)^3} = \frac{-4x+6y}{(2x+3y)^3}$
(8) $f_x = 2x\cos(x^2 - y^2)$, $f_{xx} = 2\cos(x^2 - y^2) - 4x^2\sin(x^2 - y^2)$
(9) $f_x = yz^2e^{xyz^2}$, $f_{xy} = z^2e^{xyz^2} + (yz^2)\cdot(xz^2)\cdot e^{xyz^2} = (z^2 + xyz^4)e^{xyz^2}$, and $f_{xyz} = (2z + 4xyz^3)e^{xyz^2} + (z^2 + xyz^4)\cdot(2xyz)\cdot e^{xyz^2} = 2z(x^2y^2z^4 + 3xyz^2 + 1)e^{xyz^2}$
(10) $f_x = 4\cos(4x - 3y)$, $f_{yx} = 12\sin(4x - 3y)$, $f_{xyx} = 48\cos(4x - 3y)$
(11) Since $(e^{x+y})_{xyz} = (e^{x+y})_{zxy} = ((e^{x+y})_z)_{xy} = 0$ and $(e^{\sin(y-z)})_{xyz} = ((e^{\sin(y-z)})_x)_{yz} = 0$, we get $f_{xyz} = 0$.

□

Exercise 2. Find all the second partial derivatives.

(1) $f(x,y) = \frac{x+y}{y^2-x}$
(2) $f(x,y) = 2e^{x^2+y^2}$
(3) $f(x,y,z) = 6e^x + 3xy - ye^{\sin(z)}$

Solution.

(1) We have $f_x = \frac{1}{y^2-x} - \frac{(x+y)\cdot(-1)}{(y^2-x)^2} = \frac{y^2+y}{(y^2-x)^2}$ and $f_y = \frac{1}{y^2-x} - \frac{(x+y)\cdot(2y)}{(y^2-x)^2} = \frac{-x-2yx-y^2}{(y^2-x)^2}$. So

$$f_{xx} = \frac{2y(y+1)}{(y^2-x)^3}$$

$$f_{xy} = f_{yx} = \frac{2y+1}{(y^2-x)^2} - 2\frac{(y^2+y)\cdot(2y)}{(y^2-x)^3} = \frac{-2xy - x - (2y+3)y^2}{(y^2-x)^3}$$

$$f_{yy} = -\frac{2x+2y}{(y^2-x)^2} + 2\frac{(x+2yx+y^2)\cdot(2y)}{(y^2-x)^3} = \frac{2x^2+6x(y+1)y+2y^3}{(y^2-x)^3}$$

(2) We have $f_x = 4xe^{x^2+y^2}$, $f_y = 4ye^{x^2+y^2}$. So

$$f_{xx} = 4e^{x^2+y^2} + 8x^2e^{x^2+y^2}$$

$$f_{xy} = f_{yx} = 8xye^{x^2+y^2}$$

$$f_{yy} = 4e^{x^2+y^2} + 8y^2e^{x^2+y^2}$$

(3) We have $f_x = 6e^x + 3y$, $f_y = 3x - e^{\sin(z)}$, $f_z = -y\cos(z)e^{\sin(z)}$. So

$$f_{xx} = 6e^x$$

$$f_{xy} = f_{yx} = 3$$

$$f_{xz} = f_{zx} = 0$$

$$f_{yy} = 0$$

$$f_{yz} = f_{zy} = -\cos(z)e^{\sin(z)}$$

$$f_{zz} = y\sin(z)e^{\sin(z)} - y\cos^2(z)e^{\sin(z)}$$

□

Exercise 3. Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

(1) $x^2 - y^2 + z^2 - 2z = 4$
(2) $yz + x\ln y = z^2$
(3) $e^z = xyz$

Solution.

(1) Take $\frac{\partial}{\partial x}$ on both sides to get

$$2x + 2z\frac{\partial z}{\partial x} - 2\frac{\partial z}{\partial x} = 0$$

or

$$(2z-2)\frac{\partial z}{\partial x} = -2x$$

or

$$\frac{\partial z}{\partial x} = -\frac{x}{z-1}$$

Take $\frac{\partial}{\partial y}$ on both sides to get

$$-2y + 2z\frac{\partial z}{\partial y} - 2\frac{\partial z}{\partial y} = 0$$

or

$$(2z-2)\frac{\partial z}{\partial y} = 2y$$

or

$$\frac{\partial z}{\partial y} = \frac{y}{z-1}$$

(2) Take $\frac{\partial}{\partial x}$ on both sides to get

$$y\frac{\partial z}{\partial x} + \ln y = 2z\frac{\partial z}{\partial x}$$

or

$$(2z-y)\frac{\partial z}{\partial x} = \ln y$$

or

$$\frac{\partial z}{\partial x} = \frac{\ln y}{2z-y}$$

Take $\frac{\partial}{\partial y}$ on both sides to get

$$z + y\frac{\partial z}{\partial y} + \frac{x}{y} = 2z\frac{\partial z}{\partial y}$$

or

$$z + \frac{x}{y} = (2z-y)\frac{\partial z}{\partial y}$$

or

$$\frac{\partial z}{\partial y} = \frac{z + \frac{x}{y}}{2z-y}$$

(3) Take $\frac{\partial}{\partial x}$ on both sides to get

$$\frac{\partial z}{\partial x}e^z = yz + xy\frac{\partial z}{\partial x}$$

or

$$\frac{\partial z}{\partial x}(e^z - xy) = yz$$

or

$$\frac{\partial z}{\partial x} = \frac{yz}{e^z - xy}$$

Take $\frac{\partial}{\partial y}$ on both sides to get

$$\frac{\partial z}{\partial y}e^z = xz + xy\frac{\partial z}{\partial y}$$

or

$$\frac{\partial z}{\partial y}(e^z - xy) = xz$$

or

$$\frac{\partial z}{\partial y} = \frac{xz}{e^z - xy}$$

□

Exercise 4. Use implicit differentiation to find $\frac{\partial^2 z}{\partial x \partial y}$.

(1) $x^2 - y^2 + z^2 - 2z = 4$
(2) $yz + x \ln y = z^2$
(3) $e^z = xyz$

Solution. We had computed $\frac{\partial z}{\partial x}$ above, so we need to compute its $\frac{\partial}{\partial y}$.

(1) We have

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{x}{z-1}\right) = \frac{x}{(z-1)^2}\frac{\partial z}{\partial y} = \frac{x}{(z-1)^2}\frac{y}{z-1} = \frac{xy}{(z-1)^3}$$

(2) We have

$$\begin{aligned}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\ln y}{2z-y}\right) &= \frac{\frac{1}{y}}{2z-y} - \frac{\ln y}{(2z-y)^2}\left(2\frac{\partial z}{\partial y} - 1\right) \\ &= \frac{1}{y(2z-y)^2} - \frac{\ln y}{(2z-y)^2}\left(\frac{2z+\frac{2x}{y}}{2z-y} - 1\right)\end{aligned}$$

(3) We have

$$\begin{aligned}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{yz}{e^z - xy}\right) &= \frac{z + y\frac{\partial z}{\partial y}}{e^z - xy} - \frac{yz}{(e^z-xy)^2}\left(e^z\frac{\partial z}{\partial y} - x\right) \\ &= \frac{z + y\frac{xz}{e^z-xy}}{e^z - xy} - \frac{yz}{(e^z-xy)^2}\left(e^z\frac{xz}{e^z-xy} - x\right)\end{aligned}$$

□

13. Tangent planes and approximations

Exercise 1. Find an equation of the tangent plane of the given surface at the specified point.

(1) The graph of $f(x,y) = x^2 + y^2 + x - 2y$, at $x = 0, y = 1$.
(2) The graph of $f(x,y) = y^2 e^x$, at $(0,3,9)$.
(3) The graph of $f(x,y) = \frac{x}{y^2}$, at $(-4,2,-1)$.
(4) The graph of $f(x,y) = x\sin(x+y)$, at $(-1,1,0)$.
(5) The surface $x^2 + 4y^2 + 9z^2 = 1$, at $(\frac{3}{7}, \frac{1}{7}, \frac{2}{7})$.
(6) The surface $xy + yz + zx = 0$, at $(-1,2,2)$.
(7) The surface $xe^z + yz = xy$, at $(\frac{1}{2}, -e, 1)$.

Solution. (1) We have $f(0,1) = -1$. Also,

$$f_x = 2x + 1, \quad f_y = 2y - 2,$$

so

$$f_x(0,1) = 1, \quad f_y(0,1) = 0$$

So the tangent plane has equation

$$z + 1 = f_x(0,1)(x-0) + f_y(0,1)(y-1) = x.$$

(2) We have

$$f_x = y^2 e^x, \quad f_y = 2ye^x.$$

So

$$f_x(0,3) = 9, \quad f_y(0,3) = 6.$$

So the tangent plane has equation

$$z - 9 = f_x(0,3)(x-0) + f_y(0,3)(y-3) = 9x + 6(y-3) = 9x + 6y - 18,$$

or $z = 9x + 6y - 9$.

(3) We have

$$f_x = \frac{1}{y^2}, \quad f_y = -2\frac{x}{y^3}.$$

So

$$f_x(-4,2) = \frac{1}{4}, \quad f_y(-4,2) = 1.$$

So the tangent plane has equation

$$z + 1 = f_x(-4,2)(x+4) + f_y(-4,2)(y-2) = \frac{1}{4}(x+4) + (y-2) = \frac{x}{4} + y - 1,$$

or $z = \frac{x}{4} + y - 2$.

(4) We have

$$f_x = \sin(x+y) + x\cos(x+y), \quad f_y = x\cos(x+y),$$

so

$$f_x(-1,1) = -1, \quad f_y(-1,1) = -1.$$

So the tangent plane has equation

$$z = f_x(-1,1)(x+1) + f_y(-1,1)(y-1) = -(x+1) - (y-1) = -x - y.$$

(5) To find the tangent plane, we need to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. Note that by taking $\frac{\partial}{\partial x}$ on the implicit equation we get

$$2x + 18z\frac{\partial z}{\partial x} = 0,$$

or $\frac{\partial z}{\partial x} = -\frac{x}{9z}$, and by taking $\frac{\partial}{\partial y}$ on the implicit equation we get

$$8y + 18z\frac{\partial z}{\partial y} = 0,$$

or $\frac{\partial z}{\partial y} = -\frac{4y}{9z}$. So, at $(\frac{3}{7}, \frac{1}{7}, \frac{2}{7})$, we have

$$\frac{\partial z}{\partial x}(\frac{3}{7}, \frac{1}{7}) = -\frac{\frac{3}{7}}{\frac{18}{7}} = -\frac{1}{6},$$

$$\frac{\partial z}{\partial y}(\frac{3}{7}, \frac{1}{7}) = -\frac{\frac{4}{7}}{\frac{18}{7}} = -\frac{2}{9}.$$

So, the tangent plane has equation

$$z - \frac{2}{7} = \frac{\partial z}{\partial x}(\frac{3}{7}, \frac{1}{7})(x - \frac{3}{7}) + \frac{\partial z}{\partial y}(\frac{3}{7}, \frac{1}{7})(y - \frac{1}{7}) = -\frac{1}{6}(x - \frac{3}{7}) - \frac{2}{9}(y - \frac{1}{7}) = -\frac{1}{6}x - \frac{2}{9}y + \frac{13}{126},$$

or

$$z = -\frac{1}{6}x - \frac{2}{9} + \frac{49}{126} = -\frac{1}{6}x - \frac{2}{9} + \frac{7}{18}.$$

(6) To find the tangent plane, we need to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. Note that by taking $\frac{\partial}{\partial x}$ on the implicit equation we get

$$y + y\frac{\partial z}{\partial x} + \frac{\partial z}{\partial x}x + z = 0,$$

or

$$(x+y)\frac{\partial z}{\partial x} = -y - z,$$

or

$$\frac{\partial z}{\partial x} = -\frac{y+z}{x+y}.$$

By taking $\frac{\partial}{\partial y}$ on the implicit equation we get

$$x + z + y\frac{\partial z}{\partial y} + \frac{\partial z}{\partial y}x = 0,$$

or

$$(x+y)\frac{\partial z}{\partial y} = -x - z,$$

or

$$\frac{\partial z}{\partial y} = -\frac{x+z}{x+y}.$$

So, at $(-1, 2, 2)$, we have

$$\frac{\partial z}{\partial x}(-1,2) = -4, \quad \frac{\partial z}{\partial y}(-1,2) = -1.$$

So the tangent plane has equation

$$z - 2 = \frac{\partial z}{\partial x}(-1,2)(x+1) + \frac{\partial z}{\partial y}(-1,2)(y-2) = -4(x+1) - (y-2) = -4x - y - 2,$$

or

$$z = -4x - y.$$

(7) To find the tangent plane, we need to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. Note that by taking $\frac{\partial}{\partial x}$ on the implicit equation we get

$$e^z + xe^z\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial x} = y,$$

or

$$(xe^z + y)\frac{\partial z}{\partial x} = y - e^z,$$

or

$$\frac{\partial z}{\partial x} = \frac{y - e^z}{xe^z + y}.$$

By taking $\frac{\partial}{\partial y}$ on the implicit equation we get

$$xe^z\frac{\partial z}{\partial y} + z + y\frac{\partial z}{\partial y} = x,$$

or

$$(xe^z + y)\frac{\partial z}{\partial y} = x - z,$$

or

$$\frac{\partial z}{\partial y} = \frac{x - z}{xe^z + y}.$$

So, at $(\frac{1}{2}, -e, 1)$, we have

$$\frac{\partial z}{\partial x}(\frac{1}{2}, -e) = \frac{-e-e}{\frac{1}{2}e - e} = 4, \quad \frac{\partial z}{\partial y}(\frac{1}{2}, -e) = \frac{\frac{1}{2}-1}{\frac{1}{2}e - e} = \frac{1}{e}.$$

So the tangent plane has equation

$$z - 1 = \frac{\partial z}{\partial x}(\frac{1}{2}, -e)(x - \frac{1}{2}) + \frac{\partial z}{\partial y}(\frac{1}{2}, -e)(y + e) = 4(x - \frac{1}{2}) + \frac{1}{e}(y + e) = 4x + \frac{y}{e} - 1,$$

or

$$z = 4x + \frac{y}{e}.$$

□

Exercise 2. Approximate the number.

(1) $f(0.01, 0.01)$, for $f(x, y) = \sin(x + 2y)$.
(2) $f(0.1, -0.1)$, for $f(x, y) = e^{y\cos(x)}$.
(3) $f(\frac{\pi}{2} + 0.02, 0.01)$, for $f(x, y) = xy\sin(x + y)$.
(4) $f(3.02, 1.97, 5.99)$, for $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.
(5) $f(1.02, 0.01, 1.02)$, for $f(x, y, z) = z^2 \ln(x^2 - y^2)$.

Solution. (1) Note

$$f(0.01, 0.01) \sim f(0, 0) + f_x(0, 0) \cdot 0.01 + f_y(0, 0) \cdot 0.01$$

Since $f(0, 0) = 0$, $f_x = \cos(x + 2y)$, $f_x(0, 0) = 1$, $f_y = 2\cos(x + 2y)$, $f_y(0, 0) = 2$, we have

$$f(0.01) \sim 0.01 + 0.02 = 0.03.$$

(2) Note

$$f(0.1, -0.1) \sim f(0, 0) + f_x(0, 0) \cdot 0.1 + f_y(0, 0) \cdot (-0.1)$$

Since $f(0, 0) = 1$, $f_x = -e^{y\cos(x)} y \sin(x)$, $f_x(0, 0) = 0$, $f_y = e^{y\cos(x)} \cos(x)$, $f_y(0, 0) = 1$, we have

$$f(0.1, -0.1) \sim 1 - 0.1 = 0.9.$$

(3) Note

$$f(\frac{\pi}{2} + 0.02, 0.01) \sim f(\frac{\pi}{2}, 0) + f_x(\frac{\pi}{2}, 0) \cdot 0.02 + f_y(\frac{\pi}{2}, 0) \cdot 0.01$$

Since $f(\frac{\pi}{2}, 0) = 0$, $f_x = y\sin(x + y) + xy\cos(x + y)$, $f_x(\frac{\pi}{2}, 0) = 0$, $f_y = x\sin(x + y) + xy\cos(x + y)$, $f_y(\frac{\pi}{2}, 0) = \frac{\pi}{2}$, we have

$$f(\frac{\pi}{2} + 0.02, 0.01) \sim \frac{\pi}{200}.$$

(4) Note

$$f(3.02, 1.97, 5.99) \sim f(3, 2, 6) + f_x(3, 2, 6) \cdot 0.02 + f_y(3, 2, 6) \cdot (-0.03) + f_z(3, 2, 6) \cdot (-0.01)$$

Since $f(3, 2, 6) = 7$, $f_x(x, y, z) = \frac{x}{\sqrt{x^2+y^2+z^2}}$, $f_x(3, 2, 6) = \frac{3}{7}$, $f_y(x, y, z) = \frac{y}{\sqrt{x^2+y^2+z^2}}$, $f_y(3, 2, 6) = \frac{2}{7}$, $f_z(x, y, z) = \frac{z}{\sqrt{x^2+y^2+z^2}}$, $f_z(3, 2, 6) = \frac{6}{7}$. So

$$f(3.02, 1.97, 5.99) \sim 7 + \frac{6}{700} - \frac{6}{700} - \frac{6}{700} = 7 - \frac{6}{700}.$$

(5) Note

$$f(1.02, 0.01, 1.02) \sim f(1,0,1) + f_x(1,0,1)\cdot 0.02 + f_y(1,0,1)\cdot 0.01 + f_z(1,0,1)\cdot 0.02$$

Since $f(1,0,1) = 0$, $f_x = \frac{2xz^2}{x^2-y^2}$, $f_x(1,0,1) = 2$, $f_y = \frac{-2yz^2}{x^2-y^2}$, $f_y(1,0,1) = 0$, $f_z = 2z\ln(x^2-y^2)$, $f_z(1,0,1) = 0$, we have

$$f(1.02, 0.01, 1.02) \sim 0.04.$$

□

Exercise 3. Find the differential Δf.

(1) $f(x,y) = x^2 - xy + 3y^2$.
(2) $f(x,y) = 1 + x\ln(xy-5)$.
(3) $f(x,y,z) = xze^{-y^2-z^2}$.

Solution. (1) Note that $f_x = 2x - y$, $f_y = -x + 6y$, so

$$\Delta f = (2x-y)\Delta x + (-x+6y)\Delta y.$$

(2) Note that $f_x = \ln(xy-5) + x\frac{y}{xy-5}$, $f_y = x\frac{x}{xy-5}$, so

$$\Delta f = \left(\ln(xy-5) + \frac{xy}{xy-5}\right)\Delta x + \frac{x^2}{xy-5}\Delta y.$$

(3) Note that $f_x = ze^{-y^2-z^2}$, $f_y = -2xyze^{-y^2-z^2}$, $f_z = xe^{-y^2-z^2} + xze^{-y^2-z^2}(-2z) = (x-2xz^2)e^{-y^2-z^2}$, so

$$\Delta f = ze^{-y^2-z^2}\Delta x - 2xyze^{-y^2-z^2}\Delta y + (x-2xz^2)e^{-y^2-z^2}\Delta z$$

□

Exercise 4. Find the margin of error of the quantity.

(1) $f(x,y) = xe^{xy}$, with $x = 2\pm 0.1$ and $y = 0 \pm 0.2$.
(2) $f(x,y) = x\ln(y+1)$, with $x = -1 \pm 0.1$ and $y = 0 \pm 0.1$.
(3) The volume of the box with sides $(2\pm 0.05)\times(10\pm 0.07)\times(10\pm 0.02)$.
(4) $f(x,y,z) = x^2y^4 + y^3z^5$, with $x = 3\pm 0.01$, $y = 2\pm 0.01$ and $z = 1 \pm 0.01$.

Solution. (1) Note that $f_x = e^{xy} + xye^{xy}$, and $f_y = x^2e^{xy}$, so $f_x(2,0) = 1$ and $f_y(2,0) = 4$. So

$$\mathrm{MOE}_f = |f_x(2,0)|\,\mathrm{MOE}_x + |f_y(2,0)|\,\mathrm{MOE}_y = 0.1 + 4\cdot 0.2 = 0.9.$$

(2) Note that $f_x = \ln(y+1)$ and $f_y = \frac{x}{y+1}$, so $f_x(-1,0) = 0$ and $f_y(-1,0) = -1$. So

$$\mathrm{MOE}_f = |f_x(-1,0)|\,\mathrm{MOE}_x + |f_y(-1,0)|\,\mathrm{MOE}_y = 0.1.$$

(3) Note that the volume $V(x,y,z) = xyz$, for a box with three sides x, y, z. Then $V_x = yz$, $V_y = xz$, $V_z = xy$, so $V_x(2,10,10) = 100$, $V_y(2,10,10) = 20$, $V_z(2,10,10) = 20$. So

$$\mathrm{MOE}_V = |V_x(2,10,10)|\,\mathrm{MOE}_x + |V_y(2,10,10)|\,\mathrm{MOE}_y + |V_z(2,10,10)|\,\mathrm{MOE}_z = 5 + 1.4 + 0.4 = 6.8.$$

(4) Note that $f_x = 2xy^4$, $f_y = 4x^2y^3 + 3y^2z^5$, $f_z = 5y^3z^4$, so $f_x(3,2,1) = 96$, $f_y(3,2,1) = 300$, $f_z(3,2,1) = 40$. So

$$\mathrm{MOE}_f = |f_x(3,2,1)|\,\mathrm{MOE}_x + |f_y(3,2,1)|\,\mathrm{MOE}_y + |f_z(3,2,1)|\,\mathrm{MOE}_z = 0.96 + 3 + 0.4 = 4.36.$$

□

Exercise 5. Approximate using the table of values.

(1) The wave heights, when the wind has been blowing for 32 hours at 43 knots. The following is the table of wave heights in the open sea, measured in feet.

Duration (hours)

Wind speed (knots)	v \ t	5	10	15	20	30	40	50
	20	5	7	8	8	9	9	9
	30	9	13	16	17	18	19	19
	40	14	21	25	28	31	33	33
	50	19	29	36	40	45	48	50
	60	24	37	47	54	62	67	69

(2) The perceived temperature, when the actual temperature is -17°C and the wind speed is 52km/h. The following is the table of perceived temperature, measured in °C.

Wind speed (km/h)

Actual temperature (°C)	T \ v	20	30	40	50	60	70
	-10	-18	-20	-21	-22	-23	-23
	-15	-24	-26	-27	-29	-30	-30
	-20	-30	-33	-34	-35	-36	-37
	-25	-37	-39	-41	-42	-43	-44

(3) The surface area of a human body, when the weight is 177 pounds and the height is 5.7 feet. The following is the table of the surface area of a human body, measured in square feet.

Weight (pounds)

Height (feet)	h \ w	150	160	170	180	190	200
	5	17.9	18.4	18.8	19.3	19.7	20.2
	5.5	19.1	19.7	20.2	20.7	21.2	21.6
	6	20.4	20.9	21.5	22	22.5	23
	6.5	21.6	22.2	22.8	23.3	23.9	24.4

Solution. (1) Let $h(v,t)$ be the wave height (in feet) given the wind speed v (in knots) and the duration t (in hours). We are interested in $h(43,32)$. We have

$$h(43,32) \sim h(40,30) + 3h_v(40,30) + 2h_t(40,30).$$

We have

$$h_v(40,30) \sim \frac{h(50,30)-h(40,30)}{10} = \frac{45-31}{10} = 1.4,$$
$$h_t(40,30) \sim \frac{h(40,40)-h(40,30)}{10} = \frac{33-31}{10} = 0.2,$$

so

$$h(43,32) \sim 31 + 4.2 + 0.4 = 35.6.$$

(2) Let $I(T,v)$ be the perceived tempertaure (in °C) given the actual temperature T (in °C) and the wind speed v (in km/h). We are interested in $I(-17,52)$. We have

$$I(-17,52) \sim I(-15,50) - 2I_T(-15,50) + 2I_v(-15,50).$$

We have

$$I_T(-15,50) \sim \frac{I(-10,50)-I(-15,50)}{5} = \frac{-22-(-29)}{5} = 1.4,$$

$$I_v(-15,50) \sim \frac{I(-15,60)-I(-15,50)}{10} = \frac{-30-(-29)}{10} = -0.1,$$

so

$$I(-17,52) \sim -29 - 2.8 + 0.2 = -31.6.$$

(3) Let $S(w,h)$ be the surface area of a human body (in square feet) given the weight w (in pounds) and the height h (in feet). We are interested in $S(177,5.7)$. We have

$$S(177,5.7) \sim S(180,5.5) - 3S_w(180,5.5) + 0.2S_h(180,5.5)$$

We have

$$S_w(180,5.5) \sim \frac{S(190,5.5)-S(180,5.5)}{10} = \frac{21.2-20.7}{10} = 0.05,$$

$$S_h(180,5.5) \sim \frac{S(180,6)-S(180,5.5)}{0.5} = \frac{22-20.7}{0.5} = 2.6,$$

so

$$S(177,5.7) \sim 20.7 - 0.15 + 0.52 = 21.07$$

□

14. Chain rule

Exercise 1. Find the derivative(s).

(1) $\frac{d}{dt}f(x(t),y(t))$, where

$$f(x,y) = xy^2, \quad x(t) = t^3, \quad y(t) = \frac{1}{t}$$

(2) $\frac{d}{dt}f(x(t),y(t))$, where

$$f(x,y) = e^{xy}, \quad x(t) = \ln(t), \quad y(t) = t$$

(3) $\frac{d}{dt}f(x(t),y(t))$, where

$$f(x,y) = x^3 + xy^2, \quad x(t) = \sin(t), \quad y(t) = \cos(t)$$

(4) $\frac{d}{dt}f(x(t),y(t),z(t))$, where

$$f(x,y,z) = xy + yz, \quad x(t) = t^2 - 1, \quad y(t) = 2t, \quad z(t) = t^3$$

(5) $\frac{d}{dt}f(x(t),y(t),z(t))$, where

$$f(x,y,z) = x^2yz, \quad x(t) = 2t, \quad y(t) = t^3 + t, \quad z(t) = t^2 - 1$$

(6) $\frac{d}{dt}f(x(t),y(t),z(t))$, where

$$f(x,y,z) = \sqrt{x+yz}, \quad x(t) = \sin^2(t), \quad y(t) = \cos(t), \quad z(t) = \cos(t)$$

(7) $\frac{d}{dt}f(x(t),y(t),z(t))$, where

$$f(x,y,z) = \ln(1+xyz), \quad x(t) = t^2, \quad y(t) = \frac{1}{t}, \quad z(t) = e^t$$

(8) $\frac{\partial}{\partial s} f(x(s,t), y(s,t))$ and $\frac{\partial}{\partial t} f(x(s,t), y(s,t))$, where

$$f(x,y) = x^2 + xy, \quad x(s,t) = s + t, \quad y(s,t) = st$$

(9) $\frac{\partial}{\partial s} f(x(s,t), y(s,t))$ and $\frac{\partial}{\partial t} f(x(s,t), y(s,t))$, where

$$f(x,y) = xe^{xy}, \quad x(s,t) = s^2 t, \quad y(s,t) = s - t$$

(10) $\frac{\partial}{\partial s} f(x(s,t), y(s,t))$ and $\frac{\partial}{\partial t} f(x(s,t), y(s,t))$, where

$$f(x,y) = \sin(\ln(x^2 y)), \quad x(s,t) = e^{st}, \quad y(s,t) = e^{st^2}$$

Solution. (1) We have $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$. Since

$$\frac{\partial f}{\partial x} = y^2, \quad \frac{\partial f}{\partial y} = 2xy,$$

$$\frac{dx}{dt} = 3t^2, \quad \frac{dy}{dt} = -\frac{1}{t^2}$$

we have

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = y^2 \cdot (3t^2) + (2xy) \cdot (-\frac{1}{t^2}) = 3 - 2 = 1$$

(2) We have $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$. Since

$$\frac{\partial f}{\partial x} = ye^{xy}, \quad \frac{\partial f}{\partial y} = xe^{xy},$$

$$\frac{dx}{dt} = \frac{1}{t}, \quad \frac{dy}{dt} = 1$$

we have

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = ye^{xy} \cdot \frac{1}{t} + xe^{xy} = e^{t\ln(t)} + \ln(t)e^{t\ln t} = t^t(1 + \ln(t))$$

(3) We have $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$. Since

$$\frac{\partial f}{\partial x} = 3x^2 + y^2, \quad \frac{\partial f}{\partial y} = 2xy,$$

$$\frac{dx}{dt} = \cos(t), \quad \frac{dy}{dt} = -\sin(t)$$

we have

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = (3x^2 + y^2)\cos(t) - 2xy\sin(t)$$

$$= (3\sin^2 t + \cos^2 t)\cos t - 2\sin^2 t \cos t = (\sin^2 t + \cos^2 t)\cos t = \cos t$$

(4) We have $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$. Since

$$\frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x + z, \quad \frac{\partial f}{\partial z} = y$$

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 2, \quad \frac{dz}{dt} = 3t^2$$

we have

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 2yt + 2(x+z) + 3yt^2 = 4t^2 + 2(t^3+t^2-1) + 6t^3 = 8t^3 + 6t^2 - 2$$

(5) We have $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$. Since

$$\frac{\partial f}{\partial x} = 2xyz, \quad \frac{\partial f}{\partial y} = x^2 z, \quad \frac{\partial f}{\partial z} = x^2 y$$

$$\frac{dx}{dt} = 2, \quad \frac{dy}{dt} = 3t^2 + 1, \quad \frac{dz}{dt} = 2t$$

we have

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 4xyz + (3t^2+1)x^2 z + 2tx^2 y$$

$$= 4 \cdot (2t) \cdot (t^3+t) \cdot (t^2-1) + (3t^2+1) \cdot (2t)^2 \cdot (t^2-1) + 2t \cdot (2t)^2 \cdot (t^3+t)$$

$$= 8t^2(t^4-1) + 4t^2(3t^4 - 2t^2 - 1) + 8t^3(t^3+t) = 28t^6 - 12t^2$$

(6) We have $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$. Since

$$\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{x+yz}}, \quad \frac{\partial f}{\partial y} = \frac{z}{2\sqrt{x+yz}}, \quad \frac{\partial f}{\partial z} = \frac{y}{2\sqrt{x+yz}}$$

$$\frac{dx}{dt} = 2\sin(t)\cos(t), \quad \frac{dy}{dt} = -\sin(t), \quad \frac{dz}{dt} = -\sin(t)$$

we have

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = \frac{\sin(t)\cos(t)}{\sqrt{x+yz}} - \frac{\sin(t)z}{2\sqrt{x+yz}} - \frac{\sin(t)y}{2\sqrt{x+yz}} = \frac{\sin(t)\cos(t) - \sin(t)\cos(t)}{\sqrt{x+yz}} = 0$$

(7) We have $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$. Since

$$\frac{\partial f}{\partial x} = \frac{yz}{1+xyz}, \quad \frac{\partial f}{\partial y} = \frac{xz}{1+xyz}, \quad \frac{\partial f}{\partial z} = \frac{xy}{1+xyz}$$

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = -\frac{1}{t^2}, \quad \frac{dz}{dt} = e^t$$

we have

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = \frac{2tyz}{1+xyz} - \frac{xz}{t^2(1+xyz)} + \frac{e^t xy}{1+xyz}$$

$$= \frac{2e^t}{1+te^t} - \frac{t^2 e^t}{t^2(1+te^t)} + \frac{te^t}{1+te^t} = \frac{e^t + te^t}{1+te^t}$$

(8) We have $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$ and $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$. Since

$$\frac{\partial f}{\partial x} = 2x + y, \quad \frac{\partial f}{\partial y} = x,$$

$$\frac{\partial x}{\partial s} = 1, \quad \frac{\partial y}{\partial s} = t,$$

$$\frac{\partial x}{\partial t} =, 1 \quad \frac{\partial y}{\partial t} = s,$$

we have

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = 2x + y + xt = 2(s+t) + st + (s+t)t = 2st + t^2 + 2s + 2t$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = 2x + y + xs = 2(s+t) + st + (s+t)s = 2st + s^2 + 2s + 2t$$

(9) We have $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$ and $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$. Since

$$\frac{\partial f}{\partial x} = e^{xy} + xye^{xy}, \quad \frac{\partial f}{\partial y} = x^2 e^{xy},$$

$$\frac{\partial x}{\partial s} = 2st, \quad \frac{\partial y}{\partial s} = 1,$$

$$\frac{\partial x}{\partial t} = s^2, \quad \frac{\partial y}{\partial t} = -1,$$

we have

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = (e^{xy} + xye^{xy}) \cdot (2st) + x^2 e^{xy} = st(2 + 2s^2 t(s-t) + s^3 t)e^{s^2 t(s-t)}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = (e^{xy} + xye^{xy}) \cdot s^2 - x^2 e^{xy} = s^2(1 + s^2 t(s-2t))e^{s^2 t(s-t)}$$

(10) We have $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$ and $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$. Since

$$\frac{\partial f}{\partial x} = \cos(\ln(x^2 y))\frac{2xy}{x^2 y} = \frac{2\cos(\ln(x^2 y))}{x}, \quad \frac{\partial f}{\partial y} = \cos(\ln(x^2 y))\frac{x^2}{x^2 y} = \frac{\cos(\ln(x^2 y))}{y},$$

$$\frac{\partial x}{\partial s} = te^{st}, \quad \frac{\partial y}{\partial s} = t^2 e^{st^2},$$

$$\frac{\partial x}{\partial t} = se^{st}, \quad \frac{\partial y}{\partial t} = 2ste^{st^2},$$

we have

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \frac{2\cos(\ln(x^2 y))}{x} te^{st} + \frac{\cos(\ln(x^2 y))}{y} t^2 e^{st^2}$$

$$= 2t\cos(2st + st^2) + t^2\cos(2st + st^2)$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{2\cos(\ln(x^2 y))}{x} se^{st} + \frac{\cos(\ln(x^2 y))}{y} 2ste^{st^2}$$

$$= 2s\cos(2st + st^2) + 2st\cos(2st + st^2)$$

□

Exercise 2. Find the critical points.

(1) $f(x(t), y(t))$, where

$$f(x,y) = xye^y, \quad x(t) = t-2, \quad y(t) = t$$

(2) $f(x(t), y(t))$, where

$$f(x,y) = xy^3 - x^2 y, \quad x(t) = (t+1)^2, \quad y(t) = t$$

(3) $f(x(t), y(t), z(t))$, where

$$f(x,y,z) = xy + yz, \quad x(t) = -2\sin(t^2), \quad y(t) = t^2, \quad z(t) = t^2\cos(t^2)$$

(4) $f(x(t), y(t), z(t))$, where

$$f(x,y,z) = -2xy+2z+2y^2+4x-15y+13, \quad x(t) = (t-1)^2, \quad y(t) = 2t+1, \quad z(t) = (t-1)^3$$

Solution. (1) We want to find when $\frac{df}{dt} = 0$. We have

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = ye^y + xe^y + xye^y = ((t-2)t + t - 2 + t)e^t = (t^2-2)e^t$$

So $\frac{df}{dt} = 0$ means $t^2 - 2 = 0$, or $t = \pm\sqrt{2}$.

(2) We want to find when $\frac{df}{dt} = 0$.

$$\begin{aligned}\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} &= (y^3 - 2xy)\cdot(2(t+1)) + (3xy^2 - x^2) \\ &= 2(t+1)(t^3 - 2(t+1)^2 t) + 3(t+1)^2 t^2 - (t+1)^4 \\ &= 2(t+1)(-t^3 - 4t^2 - 2t) + (t+1)^2(2t^2 - 2t - 1) \\ &= (t+1)(-2t^3 - 8t^2 - 4t + 2t^3 - 3t - 1) = (t+1)(-8t^2 - 7t - 1)\end{aligned}$$

So $\frac{df}{dt} = 0$ means $t = -1$ or $8t^2 + 7t + 1 = 0$. So, either $t = -1$ or $t = \frac{-7\pm\sqrt{17}}{16}$.

(3) We want to find when $\frac{df}{dt} = 0$.

$$\begin{aligned}\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} &= -4ty\cos(t^2) + 2t(x+z) + y(2t\cos(t^2) - 2t^3\sin(t^2)) \\ &= -4t^3\cos(t^2) + 2t(-2\sin(t^2) + t^2\cos(t^2)) + t^2(2t\cos(t^2) - 2t^3\sin(t^2)) \\ &= (-4t^3 + 2t^3 + 2t^3)\cos(t^2) + (-4t - 2t^5)\sin(t^2) = -2t(2+t^4)\sin(t^2).\end{aligned}$$

So $\frac{df}{dt} = 0$ means $t = 0$ or $\sin(t^2) = 0$, so either $t = 0$ or t^2 is an integer multiple of π, or $t = 0, \sqrt{\pi}, \sqrt{2\pi}, \sqrt{3\pi}, \cdots$.

(4) We want to find when $\frac{df}{dt} = 0$.

$$\begin{aligned}\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} &= (-2y+4)\cdot(2(t-1)) + (-2x + 4y - 15)\cdot 2 + 2\cdot 3(t-1)^2 \\ &= 2(t-1)(-4t+2) + 2(-2(t-1)^2 + 4(2t+1) - 15) + 6(t-1)^2 \\ &= -6t^2 + 24t - 24 = -6(t^2 - 4 + 4) = -6(t-2)^2\end{aligned}$$

so $\frac{df}{dt} = 0$ means $t = 2$.

□

Exercise 3. Find the distance.

(1) The distance between the point $P = (0,0)$ and the ellipse

$$x(t) = 2\cos t + \sin t, \quad y(t) = 2\cos t - \sin t$$

(2) The distance between the point $P = (0,0,0)$ and the parametric curve

$$x(t) = \ln(t), \quad y(t) = \cos t + \sin t, \quad z(t) = \cos t - \sin t$$

(3) The distance between the point $P = (1,0,0)$ and the parametric curve

$$x(t) = t^2, \quad y(t) = \sqrt{3}t, \quad z(t) = -t$$

Solution. (1) The distance between $(0,0)$ and $(x(t),y(t))$ is the function $f(x(t),y(t))$ where $f(x,y)=\sqrt{x^2+y^2}$. So we are interested in finding the global minimum of $f(x(t),y(t))$. This should be achieved at the critical points, $\frac{df}{dt}=0$. Now

$$\frac{df}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}=\frac{x}{\sqrt{x^2+y^2}}(-2\sin t+\cos t)+\frac{y}{\sqrt{x^2+y^2}}(-2\sin t-\cos t)$$

$$=\frac{(2\cos t+\sin t)(-2\sin t+\cos t)+(2\cos t-\sin t)(-2\sin t-\cos t)}{\sqrt{x^2+y^2}}$$

$$=\frac{-6\sin t\cos t}{\sqrt{x^2+y^2}}$$

So, $\frac{df}{dt}=0$ means $\sin t=0$ or $\cos t=0$. So the possible points are

$$(x,y)=(2,2),(-2,-2),(1,-1),(-1,1)$$

Among these, the closest to $(0,0)$ are $(1,-1)$ and $(-1,1)$, with distance $\sqrt{2}$.

(2) The distance between $(0,0,0)$ and $(x(t),y(t),(t))$ is the function $f(x(t),y(t),z(t))$ where $f(x,y,z)=\sqrt{x^2+y^2+z^2}$. So we are interested in finding the global minimum of $f(x(t),y(t),z(t))$. This should be achieved at the critical points, $\frac{df}{dt}=0$. Now

$$\frac{df}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}+\frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$=\frac{x}{\sqrt{x^2+y^2+z^2}}\frac{1}{t}+\frac{y}{\sqrt{x^2+y^2+z^2}}(-\sin t+\cos t)+\frac{z}{\sqrt{x^2+y^2+z^2}}(-\sin t-\cos t)$$

$$=\frac{\frac{\ln t}{t}+(\cos t+\sin t)(-\sin t+\cos t)-(\cos t-\sin t)(\sin t+\cos t)}{\sqrt{x^2+y^2+z^2}}=\frac{\ln t}{t\sqrt{x^2+y^2+z^2}}$$

So $\frac{df}{dt}=0$ means $\ln t=0$, or $t=1$. The corresponding point is $(0,\cos 1+\sin 1,\cos 1-\sin 1)$, whose distance to $(0,0,0)$ is

$$\sqrt{(\cos 1+\sin 1)^2+(\cos 1-\sin 1)^2}=\sqrt{2\cos^2 1+2\sin^2 1}=\sqrt{2}.$$

(3) The distance between $(1,0,0)$ and $(x(t),y(t),z(t))$ is the function $f(x(t),y(t),z(t))$ where $f(x,y,z)=\sqrt{(x-1)^2+y^2+z^2}$. So we are interested in finding the global minimum of $f(x(t),y(t),z(t))$. This should be achieved at the critical points, $\frac{df}{dt}=0$. Now

$$\frac{df}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}+\frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$=\frac{x-1}{\sqrt{(x-1)^2+y^2+z^2}}\cdot(2t)+\sqrt{3}\frac{y}{\sqrt{(x-1)^2+y^2+z^2}}-\frac{z}{\sqrt{(x-1)^2+y^2+z^2}}$$

$$=\frac{2t(t^2+1)}{\sqrt{(x-1)^2+y^2+z^2}}$$

so $\frac{df}{dt}=0$ means $t=0$. The corresponding point is $(0,0,0)$, so the distance between it and $(1,0,0)$ is 1.

□